Pham Kim Hung

Secrets in Inequalities

volume 1 - basic inequalities

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PHAM KIM HUNG

Secrets in Inequalities
(volume 1)

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Preface

You are now keeping in your hands this new book of elementary inequalities. "Yet another book of inequalities?" We hear you asking, and you may be right. Speaking with the author's words:

"Myriads of inequalities and inequality techniques appear nowadays in books and contests. Trying to learn all of them by heart is hopeless and useless. Alternatively, this books objective is to help you understand how inequalities work and how you can set up your own techniques on the spot, not just remember the ones you already learned. To get such a pragmatic mastery of inequalities, you surely need a comprehensive knowledge of basic inequalities at first. The goal of the first part of the book (chapters 1-8) is to lay down the foundations you will need in the second part (chapter 9), where solving problems will give you some practice. It is important to try and solve the problems by yourself as hard as you can, since only practice will develop your understanding, especially the problems in the second part. On that note, this books objective is not to present beautiful solutions to the problems, but to present such a variety of problems and techniques that will give you the best kind of practice."

It is true that there are very many books on inequalities and you have all the right to be bored and tired of them. But we tell you that this is not the case with this one. Just read the proof of Nesbitt's Inequality in the very beginning of the material, and you will understand exactly what we mean.

Now that you read it you should trust us that you will find in this book new and beautiful proofs for old inequalities and this alone can be a good reason to read it, or even just to take a quick look at it. You will find a first chapter dedicated to the classical inequalities: from AM-GM and Cauchy-Schwartz inequalities to the use of derivatives, to Chebyshev's and rearrangements' inequalities, you will find here the most important and beautiful stuff related to these classical topics. And then you have spectacular topics: you have symmetric inequalities, and inequalities with

convex functions and even a less known method of balancing coefficients. And the author would add

"You may think they are too simple to have a serious review. However, I emphasize that this review is essential in any inequalities book. Why? Because they make at least half of what you need to know in the realm of inequalities. Furthermore, really understanding them at a deep level is not easy at all. Again, this is the goal of the first part of the book, and it is the foremost goal of this book."

Every topic is described through various and numerous examples taken from many sources, especially from math contests around the world, from recent contests and recent books, or from (more or less) specialized sites on the Internet, which makes the book very lively and interesting to read for those who are involved in such activities, students and teachers from all over the world.

The author seems to be very interested in creating new inequalities: this may be seen in the whole presentation of the material, but mostly in the special chapter 2 (dedicated to this topic), or, again, in the end of the book. Every step in every proof is explained in such a manner that it seems very natural to think of; this also comes from the author's longing for a deep understanding of inequalities, longing that he passes on to the reader. Many exercises are left for those who are interested and, as a real professional solver, the author always advises us to try to find our own solution first, and only then read his one.

We will finish this introduction with the words of the author:

"Don't let the problems overwhelm you, though they are quite impressive problems, study applications of the first five basic inequalities mentioned above, plus the Abel formula, symmetric inequalities and the derivative method. Now relax with the AM-GM inequality - the foundational brick of inequalities."

Mircea Lascu, Marian Tetiva

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Also, many inequalities are collected from the famous forum of mathematics *www.mathlinks.ro*. I want to send my best thanks to the authors of the collected problems in the book and all other Mathlinks members who always inspired me in creating and solving other problems through their clever ideas. Many of the inequalities are collected from various Olympiad sources (mathematics contests all over the world) and the following mathematics magazines

- The Vietnamese mathematics and Youth Magazines (MYM).

- The Crux Mathematicorum (Crux).

Above all, let me express my deepest thanks to Pachiţariu Marius who helped me editing the book concerning both the language and also the mathematical exprimations.

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Abbreviations and Notations

Abbreviations

IMO	International Mathematical Olympiad
TST	Selection Test for IMO
APMO	Asian Pacific Mathematical Olympiad
MO	National Mathematical Olympiad
MYM	Mathematics and Youth Vietnamese Magazine
VMEO	The contest of the website *www.diendantoanhoc.net*
LHS, RHS	Left hand side, Right hand side
W.L.O.G	Without loss of generality

Notations

$\mathbb{N}$	The set of natural numbers
$\mathbb{N}^*$	The set of natural numbers except 0
$\mathbb{Z}$	The set of integers
$\mathbb{Z}^+$	The set of positive integers
$\mathbb{Q}$	The set of rational numbers
$\mathbb{R}$	The set of real numbers
$\mathbb{R}^+$	The set of positive real numbers

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Part I

The basic Inequalities

Chapter 1

AM-GM Inequality

1.1 AM-GM Inequality and Applications

Theorem 1 (AM-GM inequality). *For all positive real numbers $a_1, a_2, ..., a_n$, the following inequality holds*

$$\frac{a_1 + a_2 + ... + a_n}{n} \geq \sqrt[n]{a_1 a_2 ... a_n}.$$

Equality occurs if and only if $a_1 = a_2 = ... = a_n$.

PROOF. The inequality is clearly true for $n = 2$. If it is true for n numbers, it will be true for $2n$ numbers because

$$a_1 + a_2 + ... + a_{2n} \geq n\sqrt[n]{a_1 a_2 ... a_n} + n\sqrt[n]{a_{n+1} a_{n+2} ... a_{2n}} \geq 2n\sqrt[2n]{a_1 a_2 ... a_n},$$

Thus the inequality is true for every number n that is an exponent of 2. Suppose that the inequality is true for n numbers. We then choose

$$a_n = \frac{s}{n-1} \ ; \ s = a_1 + a_2 + ... + a_{n-1} \ ;$$

According to the inductive hypothesis, we get

$$s + \frac{s}{n-1} \geq n\sqrt[n]{\frac{a_1 a_2 ... a_{n-1} \cdot s}{n-1}} \Rightarrow s \geq (n-1)\sqrt[n-1]{a_1 a_2 ... a_{n-1}}.$$

Therefore if the inequality is true for n numbers, it will be true for $n - 1$ numbers. By induction (**Cauchy** induction), the inequality is true for every natural number n. Equality occurs if and only if $a_1 = a_2 = ... = a_n$.

∇

As a matter of fact, the **AM-GM** inequality is the most famous and wide-applied theorem. It is also indispensable in proving inequalities. Consider its strong applications through the following famous inequalities.

Proposition 1 (Nesbitt's inequality). (*a*). *Prove that for all non-negative real numbers a, b, c,*

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\ge\frac{3}{2}.$$

(*b*). *Prove that for all non-negative real numbers a, b, c, d,*

$$\frac{a}{b+c}+\frac{b}{c+d}+\frac{c}{d+a}+\frac{d}{a+b}\ge 2.$$

PROOF. (a). Consider the following expressions

$$S=\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\ ;$$
$$M=\frac{b}{b+c}+\frac{c}{c+a}+\frac{a}{a+b}\ ;$$
$$N=\frac{c}{b+c}+\frac{a}{c+a}+\frac{b}{a+b}\ ;$$

We have of course $M+N=3$. According to **AM-GM,** we get

$$M+S=\frac{a+b}{b+c}+\frac{b+c}{c+a}+\frac{c+a}{a+b}\ge 3\ ;$$
$$N+S=\frac{a+c}{b+c}+\frac{a+b}{c+a}+\frac{b+c}{a+b}\ge 3\ ;$$

Therefore $M+N+2S\ge 3$, and $2S\ge 3$, or $S\ge\frac{3}{2}$.

(b). Consider the following expressions

$$S=\frac{a}{b+c}+\frac{b}{c+d}+\frac{c}{d+a}+\frac{d}{a+b}\ ;$$
$$M=\frac{b}{b+c}+\frac{c}{c+d}+\frac{d}{d+a}+\frac{a}{a+b}\ ;$$
$$N=\frac{c}{b+c}+\frac{d}{c+d}+\frac{a}{d+a}+\frac{b}{a+b}\ ;$$

We have $M+N=4$. According to **AM-GM,** we get

$$M+S=\frac{a+b}{b+c}+\frac{b+c}{c+d}+\frac{c+d}{d+a}+\frac{d+a}{a+b}\ge 4\ ;$$
$$\begin{aligned}N+S&=\frac{a+c}{b+c}+\frac{b+d}{c+d}+\frac{a+c}{d+a}+\frac{b+d}{a+b}\\&=\frac{a+c}{b+c}+\frac{a+c}{a+d}+\frac{b+d}{c+d}+\frac{b+d}{a+b}\\&\ge\frac{4(a+c)}{a+b+c+d}+\frac{4(b+d)}{a+b+c+d}=4\ ;\end{aligned}$$

Therefore $M + N + 2S \geq 8$, and $S \geq 2$. The equality holds if $a = b = c = d$ or $a = c, b = d = 0$ or $a = c = 0, b = d$.

∇

Proposition 2 (Weighted AM-GM inequality). *Suppose that $a_1, a_2, ..., a_n$ are positive real numbers. If n non-negative real numbers $x_1, x_2, ..., x_n$ have sum 1 then*

$$a_1x_1 + a_2x_2 + ... + a_nx_n \geq a_1^{x_1}a_2^{x_2}...a_n^{x_n}.$$

SOLUTION. The proof of this inequality is entirely similar to the one for the classical **AM-GM** inequality. However, in the case $n = 2$, we need a more detailed proof (because the inequality is posed for *real* exponents). We have to prove that if $x, y \geq 0$, $x + y = 1$ and $a, b > 0$ then

$$ax + by \geq a^x b^y.$$

The most simple way to solve this one is to consider it for rational numbers x, y, then take a limit. Certainly, if x, y are rational numbers : $x = \dfrac{m}{m+n}$ and $y = \dfrac{n}{m+n}$, $(m, n \in \mathbb{N})$, the problem is true according to **AM-GM** inequality

$$ma + nb \geq (m+n)a^{\frac{m}{m+n}}b^{\frac{n}{m+n}} \Rightarrow ax + by \geq a^x b^y.$$

If x, y are real numbers, there exist two sequences of rational numbers $(r_n)_{n\geq 0}$ and $(s_n)_{n\geq 0}$ for which $r_n \longrightarrow x$, $s_n \longrightarrow y$, $r_n + s_n = 1$. Certainly

$$ar_n + bs_n \geq a^{r_n}b^{s_n}$$

or

$$ar_n + b(1 - r_n) \geq a^{r_n}b^{1-r_n}.$$

Taking the limit when $n \longrightarrow +\infty$, we have $ax + by \geq a^x b^y$.

∇

The **AM-GM** inequality is very simple; however, it plays a major part in many inequalities in Mathematics Contests. Some examples follow to help you get acquainted with this important inequality.

Example 1.1.1. *Let a, b, c be positive real numbers with sum 3. Prove that*

$$\sqrt{a} + \sqrt{b} + \sqrt{c} \geq ab + bc + ca.$$

(Russia MO 2004)

SOLUTION. Notice that

$$2(ab+bc+ca) = (a+b+c)^2 - a^2 + b^2 + c^2.$$

The inequality is then equivalent to

$$\sum_{cyc} a^2 + 2\sum_{cyc} \sqrt{a} \geq 9,$$

which is true by AM-GM because

$$\sum_{cyc} a^2 + 2\sum_{cyc} \sqrt{a} = \sum_{cyc} \left(a^2 + \sqrt{a} + \sqrt{a}\right) \geq 3\sum_{cyc} a = 9.$$

∇

Example 1.1.2. *Let x, y, z be positive real numbers such that $xyz = 1$. Prove that*

$$\frac{x^3}{(1+y)(1+z)} + \frac{y^3}{(1+z)(1+x)} + \frac{z^3}{(1+x)(1+y)} \geq \frac{3}{4}.$$

(IMO Shortlist 1998)

SOLUTION. We use AM-GM in the following form:

$$\frac{x^3}{(1+y)(1+z)} + \frac{1+y}{8} + \frac{1+z}{8} \geq \frac{3x}{4}.$$

We conclude that

$$\sum_{cyc} \frac{x^3}{(1+y)(1+z)} + \frac{1}{4}\sum_{cyc}(1+x) \geq \sum_{cyc} \frac{3x}{4}$$

$$\Rightarrow \sum_{cyc} \frac{x^3}{(1+y)(1+z)} \geq \frac{1}{4}\sum_{cyc}(2x-1) \geq \frac{3}{4}.$$

The equality holds for $x = y = z = 1$.

∇

Example 1.1.3. *Let a, b, c be positive real numbers. Prove that*

$$\left(1+\frac{x}{y}\right)\left(1+\frac{y}{z}\right)\left(1+\frac{z}{x}\right) \geq 2 + \frac{2(x+y+z)}{\sqrt[3]{xyz}}.$$

(APMO 1998)

SOLUTION. Certainly, the problem follows the inequality

$$\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\geq\frac{x+y+z}{\sqrt[3]{xyz}},$$

which is true by **AM-GM** because

$$3\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)=\left(\frac{2x}{y}+\frac{y}{z}\right)+\left(\frac{2y}{z}+\frac{z}{x}\right)+\left(\frac{2z}{x}+\frac{x}{y}\right)\geq\frac{3x}{\sqrt[3]{xyz}}+\frac{3y}{\sqrt[3]{xyz}}+\frac{3z}{\sqrt[3]{xyz}}.$$

∇

Example 1.1.4. *Let a,b,c,d be positive real numbers. Prove that*

$$16(abc+bcd+cda+dab)\leq(a+b+c+d)^4.$$

SOLUTION. Applying **AM-GM** for two numbers, we obtain

$$\begin{aligned}16(abc+bcd+cda+dab)&=16ab(c+d)+16cd(a+b)\\&\leq4(a+b)^2(c+d)+4(c+d)^2(a+b)\\&=4(a+b+c+d)(a+b)(c+d)\\&\leq(a+b+c+d)^3.\end{aligned}$$

The equality holds for $a=b=c=d$.

∇

Example 1.1.5. *Suppose that a,b,c are three side-lengths of a triangle with perimeter 3. Prove that*

$$\frac{1}{\sqrt{a+b-c}}+\frac{1}{\sqrt{b+c-a}}+\frac{1}{\sqrt{c+a-b}}\geq\frac{9}{ab+bc+ca}.$$

(Pham Kim Hung)

SOLUTION. Let $x=\sqrt{b+c-a},y=\sqrt{c+a-b},z=\sqrt{a+b-c}$. We get $x^2+y^2+z^2=3$. The inequality becomes

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\geq\frac{36}{9+x^2y^2+y^2z^2+z^2x^2}.$$

Let $m=xy,n=yz,p=zx$. The inequality above is equivalent to

$$(m+n+p)(m^2+n^2+p^2+9)\geq36\sqrt{mnp},$$

which is obvious by **AM-GM** because

$$m+n+p\geq3\sqrt[3]{mnp}\ ,\ m^2+n^2+p^2+9\geq12\sqrt[6]{mnp}.$$

∇

Example 1.1.6. *Let $a_1, a_2, ..., a_n$ be positive real numbers such that $a_i \in [0, i]$ for all $i \in \{1, 2, ..., n\}$. Prove that*

$$2^n a_1(a_1+a_2)...(a_1+a_2+...+a_n) \geq (n+1)a_1^2 a_2^2 ... a_n^2.$$

(Phan Thanh Nam)

SOLUTION. According to **AM-GM**,

$$\begin{aligned} a_1+a_2+...+a_k &= 1\cdot\left(\frac{a_1}{1}\right)+2\cdot\left(\frac{a_2}{2}\right)+...+k\cdot\left(\frac{a_k}{k}\right) \\ &\geq \frac{k(k+1)}{2}\left(\frac{a_1}{1}\right)^{\frac{2}{k(k+1)}}\cdot\left(\frac{a_2}{2}\right)^{\frac{4}{k(k+1)}}\cdot\left(\frac{a_k}{1}\right)^{\frac{2k}{k(k+1)}}. \end{aligned}$$

Multiplying these results for every $k \in \{1, 2, ..., n\}$, we obtain

$$\begin{aligned} \prod_{k=1}^{n}(a_1+a_2+...+a_k) &\geq \prod_{k=1}^{n}\left(\frac{k(k+1)}{2}\prod_{i=1}^{k}\left(\frac{a_i}{i}\right)^{\frac{2i}{k(k+1)}}\right) \\ &= \frac{n!(n+1)!}{2^n}\prod_{i=1}^{n}\left(\frac{a_i}{i}\right)^{c_i}, \end{aligned}$$

in which each exponent c_i is determined from

$$c_i = 2i\left(\frac{1}{i(i+1)}+\frac{1}{(i+1)(i+2)}+...+\frac{1}{n(n+1)}\right) = 2i\left(\frac{1}{i}-\frac{1}{n+1}\right) \leq 2.$$

Because $a_i \leq i \ \forall i \in \{1, 2, ..., n\}$, $\left(\frac{a_i}{i}\right)^{c_i} \geq \left(\frac{a_i}{i}\right)^2$ and

$$a_1(a_1+a_2)...(a_1+a_2+...+a_n) \geq \frac{n!(n+1)!}{2^n}\prod_{i=1}^{n}\left(\frac{a_i}{i}\right)^2 = \frac{n+1}{2^n}\cdot a_1^2 a_2^2...a_n^2.$$

The equality holds for $a_i = i \ \forall i \in \{1, 2, ..., n\}$.

∇

Example 1.1.7. *Let a, b, c be positive real numbers. Prove that*

$$\frac{1}{a^3+b^3+abc}+\frac{1}{b^3+c^3+abc}+\frac{1}{c^3+a^3+abc} \leq \frac{1}{abc}.$$

(USA MO 1998)

SOLUTION. Notice that $a^3+b^3 \geq ab(a+b)$, so

$$\frac{abc}{a^3+b^3+c^3} \leq \frac{abc}{ab(a+b)+abc} = \frac{c}{a+b+c}.$$

Building up two similar inequalities and adding up all of them, we have the conclusion

$$\frac{abc}{a^3+b^3+abc}+\frac{abc}{b^3+c^3+abc}+\frac{abc}{c^3+a^3+abc}\le 1.$$

Comment. Here is a similar problem from **IMO** Shortlist 1996:

★ *Consider three positive real numbers x, y, z whose product is 1. Prove that*

$$\frac{xy}{x^5+xy+y^5}+\frac{yz}{y^5+yz+z^5}+\frac{zx}{z^5+zx+x^5}\le 1.$$

∇

Example 1.1.8. *Prove that $x_1x_2...x_n \ge (n-1)^n$ if $x_1, x_2, ..., x_n > 0$ satisfy*

$$\frac{1}{1+x_1}+\frac{1}{1+x_2}+...+\frac{1}{1+x_n}=1.$$

SOLUTION. The condition implies that

$$\frac{1}{1+x_1}+\frac{1}{1+x_2}+...+\frac{1}{1+x_{n-1}}=\frac{x_n}{1+x_n}.$$

Using **AM-GM** inequality for all terms on the left hand side, we obtain

$$\frac{x_n}{1+x_n}\ge\frac{n-1}{\sqrt[n-1]{(1+x_1)(1+x_2)...(1+x_{n-1})}}.$$

Constructing n such relations for each term $x_1, x_2, ..., x_{n-1}, x_n$ and multiplying all their correlative sides, we get the desired result.

∇

Example 1.1.9. *Suppose that x, y, z are positive real numbers and $x^5+y^5+z^5=3$. Prove that*

$$\frac{x^4}{y^3}+\frac{y^4}{z^3}+\frac{z^4}{x^3}\ge 3.$$

SOLUTION. Notice that

$$(x^5+y^5+z^5)^2=x^{10}+2x^5y^5+y^{10}+2y^5z^5+z^{10}+2z^5x^5=9.$$

This form suggests the **AM-GM** inequality in the following form

$$10\cdot\frac{x^4}{y^3}+6x^5y^5+3x^{10}\ge 19x^{\frac{100}{19}}.$$

Setting up similar cyclic results and adding up all of them, we have

$$10\left(\frac{x^4}{y^3}+\frac{y^4}{z^3}+\frac{z^4}{x^3}\right)+3(x^5+y^5+z^5)^2\ge 19\left(x^{\frac{100}{19}}+y^{\frac{100}{19}}+z^{\frac{100}{19}}\right).$$

It suffices to prove that

$$x^{\frac{100}{19}} + y^{\frac{100}{19}} + z^{\frac{100}{19}} \geq x^5 + y^5 + z^5$$

which is obviously true because

$$3 + 19\sum_{cyc} x^{\frac{100}{19}} = \sum_{cyc}(1 + 19x^{\frac{100}{19}}) \geq 20\sum_{cyc} x^5.$$

∇

Example 1.1.10. *Let a, b, c be positive real numbers such that $abc = 1$. Prove that*

$$\sqrt{\frac{a+b}{a+1}} + \sqrt{\frac{b+c}{b+1}} + \sqrt{\frac{c+a}{c+1}} \geq 3.$$

(Mathlinks Contest)

SOLUTION. After applying **AM-GM** for the three terms on the left hand side expression, we only need to prove that

$$(a+b)(b+c)(c+a) \geq (a+1)(b+1)(c+1),$$

or equivalent by (because $abc = 1$)

$$ab(a+b) + bc(b+c) + ca(c+a) \geq a + b + c + ab + bc + ca.$$

According to **AM-GM**,

$$2\text{LHS} + \sum_{cyc} ab = \sum_{cyc}(a^2b + a^2b + a^2c + a^2c + bc) \geq 5\sum_{cyc} a.$$

$$2\text{LHS} + \sum_{cyc} a = \sum_{cyc}(a^2b + a^2b + b^2a + b^2a + c) \geq 5\sum_{cyc} ab.$$

Therefore

$$4\text{LHS} + 2\sum_{cyc} ab + \sum_{cyc} a \geq 5\sum_{cyc} ab + 4\sum_{cyc} ab$$

$$\Rightarrow 4\text{LHS} \geq 4\sum_{cyc} a + 4\sum_{cyc} ab = 4\text{RHS}.$$

This ends the proof. Equality holds for $a = b = c = 1$.

∇

Example 1.1.11. *Let a, b, c be the side-lengths of a triangle. Prove that*

$$(a+b-c)^a(b+c-a)^b(c+a-b)^c \leq a^a b^b c^c.$$

SOLUTION. Applying the weighted **AM-GM** inequality, we conclude that

$$\sqrt[a+b+c]{\left(\frac{a+b-c}{a}\right)^a\left(\frac{b+c-a}{b}\right)^b\left(\frac{c+a-b}{c}\right)^c}$$

$$\leq \frac{1}{a+b+c}\left(a\cdot\frac{a+b-c}{a}+b\cdot\frac{b+c-a}{b}+c\cdot\frac{c+a-b}{c}\right)=1.$$

In other words, we have

$$(a+b-c)^a(b+c-a)^b(c+a-b)^c \leq a^a b^b c^c.$$

Equality holds if and only if $a=b=c$.

∇

Example 1.1.12. *Let a,b,c be non-negative real numbers with sum 2. Prove that*

$$a^2b^2+b^2c^2+c^2a^2\leq 2.$$

SOLUTION. We certainly have

$$(ab+bc+ca)(a^2+b^2+c^2)\geq\sum_{cyc}a^3(b+c)=\sum_{cyc}ab(a^2+b^2)\geq 2\sum_{cyc}a^2b^2.$$

Applying **AM-GM**, and using $a^2+b^2+c^2+2(ab+bc+ca)=4$, we deduce that

$$2(ab+bc+ca)(a^2+b^2+c^2)\leq 4 \Rightarrow (ab+bc+ca)(a^2+b^2+c^2)\leq 2.$$

This property leads to the desired result immediately. Equality holds for $a=b=1, c=0$ up to permutation.

∇

Example 1.1.13. *Let a,b,c,d be positive real numbers. Prove that*

$$\frac{1}{a^2+ab}+\frac{1}{b^2+bc}+\frac{1}{c^2+cd}+\frac{1}{d^2+da}\geq\frac{4}{ac+bd}.$$

SOLUTION. Notice that

$$\frac{ac+bd}{a^2+ab}=\frac{a^2+ab+ac+bd}{a(a+b)}-1=\frac{a(a+c)+b(d+a)}{a(a+b)}-1=\frac{a+c}{a+b}+\frac{b(d+a)}{a(a+d)}-1.$$

According to **AM-GM** inequality, we get that

$$(ac+bd)\left(\sum_{cyc}\frac{1}{a^2+ab}\right)=\left(\sum_{cyc}\frac{a+c}{a+b}\right)+\left(\sum_{cyc}\frac{b(d+a)}{a(a+d)}\right)-4\geq\sum_{cyc}\frac{a+c}{a+b}.$$

Moreover,

$$\sum_{cyc}\frac{a+c}{a+b}=(a+c)\left(\frac{1}{a+b}+\frac{1}{c+d}\right)+(b+d)\left(\frac{1}{b+c}+\frac{1}{d+a}\right)$$
$$\geq\frac{4(a+c)}{a+b+c+d}+\frac{4(b+d)}{a+b+c+d}=4.$$

Equality holds for $a=b=c=d$.

∇

Example 1.1.14. *Let a,b,c,d,e be non-negative real numbers such that $a+b+c+d+e=5$. Prove that*

$$abc+bcd+cde+dea+eab\leq 5.$$

SOLUTION. Without loss of generality, we may assume that $e=\min(a,b,c,d,e)$. According to **AM-GM**, we have

$$abc+bcd+cde+dea+eab=e(a+c)(b+d)+bc(a+d-e)$$
$$\leq e\left(\frac{a+c+b+d}{2}\right)^2+\left(\frac{b+c+a+d-e}{3}\right)^2$$
$$=\frac{e(5-e)^2}{4}+\frac{(5-2e)^3}{27}.$$

It suffices to prove that

$$\frac{e(5-e)^2}{4}+\frac{(5-2e)^3}{27}\leq 5$$

which can be reduced to $(e-1)^2(e+8)\geq 0$.

∇

Example 1.1.15. *Let a,b,c,d be positive real numbers. Prove that*

$$\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)^2\geq\frac{1}{a^2}+\frac{4}{a^2+b^2}+\frac{9}{a^2+b^2+c^2}+\frac{16}{a^2+b^2+c^2+d^2}.$$

(Pham Kim Hung)

SOLUTION. We have to prove that

$$\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{d^2}+\sum_{sym}\frac{2}{ab}\geq\frac{4}{a^2+b^2}+\frac{9}{a^2+b^2+c^2}+\frac{16}{a^2+b^2+c^2+d^2}.$$

By **AM-GM**, we have

$$\frac{2}{ab} \geq \frac{4}{a^2+b^2};$$
$$\frac{2}{ac}+\frac{2}{bc} \geq \frac{8}{ac+bc} \geq \frac{8}{a^2+b^2+c^2};$$
$$\frac{1}{b^2}+\frac{1}{c^2} \geq \frac{4}{b^2+c^2} \geq \frac{1}{b^2+c^2+a^2};$$
$$\frac{2}{ad}+\frac{2}{bd}+\frac{2}{cd} \geq \frac{18}{ad+bd+cd} \geq \frac{16}{a^2+b^2+c^2+d^2};$$

Adding up these results, we get the conclusion immediately.

Comment. 1. By a similar approach, we can prove the similar inequality for five numbers. To do this, one needs:

$$a^2+b^2+c^2+d^2+e^2 = \left(a^2+\frac{e^2}{4}\right)+\left(b^2+\frac{e^2}{4}\right)+\left(c^2+\frac{e^2}{4}\right)+\left(d^2+\frac{e^2}{4}\right) \geq ad+bd+cd+ed.$$

2. The proof above shows the stronger inequality:

★

$$\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)^2 \geq \frac{1}{a^2}+\frac{4}{a^2+b^2}+\frac{12}{a^2+b^2+c^2}+\frac{18}{a^2+b^2+c^2+d^2}.$$

3. I conjectured the following inequality

★ *Let $a_1, a_2, ..., a_n$ be positive real numbers. Prove or disprove that*

$$\left(\frac{1}{a_1}+\frac{1}{a_2}+...+\frac{1}{a_n}\right)^2 \geq \frac{1}{a_1^2}+\frac{4}{a_1^2+a_2^2}+...+\frac{n^2}{a_1^2+a_2^2+...+a_n^2}.$$

∇

Example 1.1.16. *Determine the least M for which the inequality*

$$|ab(a^2-b^2)+bc(b^2-c^2)+ca(c^2-a^2)| \leq M(a^2+b^2+c^2)^2$$

holds for all real numbers a, b and c.

(IMO 2006, A3)

SOLUTION. Denote $x = a-b, y = b-c, z = c-a$ and $s = a+b+c$. Rewrite the inequality in the following form

$$9|sxyz| \leq M(s^2+x^2+y^2+z^2)^2$$

in which s, x, y, z are arbitrary real numbers with $x+y+z = 0$.

The fact that s is an independent variable suggests constructing a relationship between xyz and $x^2+y^2+z^2$ at first. There are two numbers, say x and y, with the same sign. Assume that $x, y \geq 0$ (the case $x, y \leq 0$ is proved similarly). By **AM-GM**, we have

$$|sxyz| = |sxy(x+y)| \leq |s| \cdot \frac{(x+y)^3}{4} \quad (1)$$

with equality for $x = y$. Let $t = x + y$. Applying **AM-GM** again, we get

$$2s^2t^6 = 2s^2 \cdot t^2 \cdot t^2 \cdot t^2 \leq \frac{(2s^2+3t^2)^4}{4^4}$$

and therefore

$$4\sqrt{2}|s|t^3 \leq \left(s^2 + \frac{3}{2}t^2\right)^2 \leq \left(s^2+x^2+y^2+z^2\right)^2 \quad (2)$$

Combining (1) and (2), we conclude

$$|sxyz| \leq \frac{1}{16\sqrt{2}}(s^2+x^2+y^2+z^2)^2.$$

This implies that $M \geq \frac{9\sqrt{2}}{32}$. To show that $M = \frac{9\sqrt{2}}{32}$ is the best constant, we need to find (s, x, y, z), or in other words, (a, b, c), for which equality holds. A simple calculation gives $(a, b, c) = \left(1 - \frac{3}{\sqrt{2}}, 1, 1 + \frac{3}{\sqrt{2}}\right)$.

∇

The most important principle when we use **AM-GM** is to choose the suitable coefficients such that equality can happen. For instance, in example 1.1.2, using **AM-GM** inequality in the following form is a common mistake (because the equality can not hold)

$$\frac{x^3}{(1+y)(1+z)} + (y+1) + (z+1) \geq 3x.$$

It's hard to give a fixed form of **AM-GM** for every problem. You depend on your own intuition but it's also helpful to look for the equality case. For example, in the above problem, guessing that the equality holds for $x = y = z = 1$, we will choose the coefficient $\frac{1}{8}$ in order to make the terms equal

$$\frac{x^3}{(1+y)(1+z)} + \frac{y+1}{8} + \frac{z+1}{8} \geq \frac{3x}{4}.$$

For problems where the equality holds for variables that are equal to each other, it seems quite easy to make couples before we use **AM-GM**. For non-symmetric

problems, this method sometimes requires a bit of flexibility (see again examples 1.1.13, 1.1.14 and 1.1.16). Sometimes you need to make a system of equations and solve it in order to find when equality holds (this method, called "balancing coefficients", will be discussed in part 6).

1.2 The Cauchy Reverse Technique

In the following section, we will connect **AM-GM** to a particular technique, called the Cauchy reverse technique. The unexpected simplicity but great effectiveness are special advantages of this technique. Warm up with the following example.

Example 1.2.1. *Let a, b, c be positive real numbers with sum 3. Prove that*

$$\frac{a}{1+b^2}+\frac{b}{1+c^2}+\frac{c}{1+a^2} \geq \frac{3}{2}.$$

(Bulgaria TST 2003)

SOLUTION. In fact, it's impossible to use **AM-GM** for the denominators because the sign will be reversed

$$\frac{a}{1+b^2}+\frac{b}{1+c^2}+\frac{c}{1+a^2} \leq \frac{a}{2b}+\frac{b}{2c}+\frac{c}{2a} \geq \frac{3}{2} \text{ ?!}$$

However, we can *use* the same application in another appearance

$$\frac{a}{1+b^2}=a-\frac{ab^2}{1+b^2} \geq a-\frac{ab^2}{2b}=a-\frac{ab}{2}.$$

The inequality becomes

$$\sum_{cyc}\frac{a}{1+b^2} \geq \sum_{cyc} a-\frac{1}{2}\sum_{cyc} ab \geq \frac{3}{2},$$

since $3\left(\sum ab\right) \leq \left(\sum a\right)^2=9$.

This ends the proof. Equality holds for $a=b=c=1$.

Comment. A similar method proves the following result

★ *Suppose that a, b, c, d are four positive real numbers with sum 4. Prove that*

$$\frac{a}{1+b^2}+\frac{b}{1+c^2}+\frac{c}{1+d^2}+\frac{d}{1+a^2} \geq 2.$$

∇

This solution seems to be magic: two apparently similar approachs of applying **AM-GM** bring about two different solutions; one is incorrect but one is correct.

So, where does this magic occur? Amazingly enough, it all comes from a simple representation of a fraction as a difference

$$\frac{a}{1+b^2}=a-\frac{ab^2}{1+b^2}.$$

With the minus sign before the new fraction $\frac{ab^2}{1+b^2}$, we can use **AM-GM** inequality in the denominator $1+b^2$ freely but we get the correct sign. This is the key feature of this impressive technique: you change a singular expression into a difference of two expressions, then estimate the second expression of this difference, which has a minus sign.

Example 1.2.2. *Suppose that a,b,c,d are four positive real numbers with sum 4. Prove that*

$$\frac{a}{1+b^2c}+\frac{b}{1+c^2d}+\frac{c}{1+d^2a}+\frac{d}{1+a^2b}\geq 2.$$

(Pham Kim Hung)

SOLUTION. According to **AM-GM**, we deduce that

$$\begin{aligned}\frac{a}{1+b^2c}=a-\frac{ab^2c}{1+b^2c}&\geq a-\frac{ab^2c}{2b\sqrt{c}}=a-\frac{ab\sqrt{c}}{2}\\ =a-\frac{b\sqrt{a\cdot ac}}{2}&\geq a-\frac{b(a+ac)}{4}.\end{aligned}$$

According to this estimation,

$$\sum_{cyc}\frac{a}{1+b^2c}\geq\sum_{cyc}a-\frac{1}{4}\sum_{cyc}ab-\frac{1}{4}\sum_{cyc}abc$$

By **AM-GM** inequality again, it's easy to refer that

$$\sum_{cyc}ab\leq\frac{1}{4}\left(\sum_{cyc}a\right)^2=4\ ;\ \sum_{cyc}abc\leq\frac{1}{16}\left(\sum_{cyc}a\right)^3=4.$$

Therefore

$$\frac{a}{1+b^2c}+\frac{b}{1+c^2d}+\frac{c}{1+d^2a}+\frac{d}{1+a^2b}\geq a+b+c+d-2=2.$$

∇

Example 1.2.3. *Let a,b,c be positive real numbers. Prove that*

$$\frac{a^3}{a^2+b^2}+\frac{b^3}{b^2+c^2}+\frac{c^3}{c^2+d^2}+\frac{d^3}{d^2+a^2}\geq\frac{a+b+c+d}{2}.$$

SOLUTION. We use the following estimation

$$\frac{a^3}{a^2+b^2}=a-\frac{ab^2}{a^2+b^2}\geq a-\frac{ab^2}{2ab}=a-\frac{b}{2}.$$

Comment. Here is a similar result for four variables

$$\frac{a^4}{a^3+2b^3}+\frac{b^4}{b^3+2c^3}+\frac{c^4}{c^3+2d^3}+\frac{d^4}{d^3+2a^3}\geq\frac{a+b+c+d}{3}.$$

∇

Example 1.2.4. *Let a,b,c be positive real numbers with sum 3. Prove that*

$$\frac{a^2}{a+2b^2}+\frac{b^2}{b+2c^2}+\frac{c^2}{c+2a^2}\geq 1.$$

SOLUTION. We use the following estimation according to **AM-GM**

$$\frac{a^2}{a+2b^2}=a-\frac{2ab^2}{a+2b^2}\geq a-\frac{2ab^2}{3\sqrt[3]{ab^4}}=a-\frac{2(ab)^{2/3}}{3},$$

which implies that

$$\sum_{cyc}\frac{a^2}{a+2b^2}\geq\sum_{cyc}a-\frac{2}{3}\sum_{cyc}(ab)^{\frac{2}{3}}.$$

It suffices to prove that

$$(ab)^{2/3}+(bc)^{2/3}+(ca)^{2/3}\leq 3.$$

By AM-GM, we have the desired result since

$$3\sum_{cyc}a\geq 2\sum_{cyc}a+\sum_{cyc}ab=\sum_{cyc}(a+b+ab)\geq 3\sum_{cyc}(ab)^{\frac{2}{3}}.$$

Comment. The inequality is still true when we change the hypothesis $a+b+c=3$ to $ab+bc+ca=3$ or even $\sqrt{a}+\sqrt{b}+\sqrt{c}=3$. (the second case is a bit more difficult). These problems are proposed to you; they will not be solved here.

∇

Example 1.2.5. *Let a,b,c be positive real numbers with sum 3. Prove that*

$$\frac{a^2}{a+2b^3}+\frac{b^2}{b+2c^3}+\frac{c^2}{c+2a^3}\geq 1.$$

SOLUTION. Using the same technique as in example 1.2.4, we only need to prove that

$$b\sqrt[3]{a^2}+c\sqrt[3]{b^2}+a\sqrt[3]{c^2}\leq 3.$$

According to **AM-GM**, we obtain

$$3\sum_{cyc} a \geq \sum_{cyc} a + 2\sum_{cyc} ab = \sum_{cyc}(a + ac + ac) \geq 3\sum_{cyc} a\sqrt[3]{c^2},$$

and the desired result follows. Equality holds for $a = b = c = 1$.

∇

Example 1.2.6. *Let a, b, c be positive real numbers which sum up to 3. Prove that*

$$\frac{a+1}{b^2+1} + \frac{b+1}{c^2+1} + \frac{c+1}{a^2+1} \geq 3.$$

SOLUTION. We use the following estimation

$$\frac{a+1}{b^2+1} = a + 1 - \frac{b^2(a+1)}{b^2+1} \geq a + 1 - \frac{b^2(a+1)}{2b} = a + 1 - \frac{ab+b}{2}.$$

Summing up the similar results for a, b, c, we deduce that

$$\sum_{cyc} \frac{a+1}{b^2+1} \geq 3 + \frac{1}{2}\sum_{cyc} a - \frac{1}{2}\sum_{cyc} ab \geq 3.$$

Comment. Here are some problems for four variables with the same appearance

★ *Let a, b, c, d be four positive real numbers with sum 4. Prove that*

$$\frac{a+1}{b^2+1} + \frac{b+1}{c^2+1} + \frac{c+1}{d^2+1} + \frac{d+1}{a^2+1} \geq 4.$$

★ *Let a, b, c, d be four positive real numbers with sum 4. Prove that*

$$\frac{1}{a^2+1} + \frac{1}{b^2+1} + \frac{1}{c^2+1} + \frac{1}{d^2+1} \geq 2.$$

∇

Example 1.2.7. *Let a, b, c be positive real numbers with sum 3. Prove that*

$$\frac{1}{1+2b^2c} + \frac{1}{1+2c^2a} + \frac{1}{1+2a^2b} \geq 1.$$

SOLUTION. We use the following estimation

$$\frac{1}{1+2b^2c} = 1 - \frac{2b^2c}{1+2b^2c} \geq 1 - \frac{2\sqrt[3]{b^2c}}{3} \geq 1 - \frac{2(2b+c)}{9}.$$

∇

Example 1.2.8. *Let a, b, c, d be non-negative real numbers with sum 4. Prove that*

$$\frac{1+ab}{1+b^2c^2} + \frac{1+bc}{1+c^2d^2} + \frac{1+cd}{1+d^2a^2} + \frac{1+da}{1+a^2b^2} \geq 4.$$

(Pham Kim Hung)

SOLUTION. Applying **AM-GM**, we have

$$\frac{1+ab}{1+b^2c^2}=(1+ab)-\frac{(1+ab)b^2c^2}{1+b^2c^2}\ge 1+ab-\frac{1}{2}(1+ab)bc.$$

Summing up similar results, we get

$$\sum_{cyc}\frac{1+ab}{1+b^2c^2}\ge 4+\sum_{cyc}ab-\frac{1}{2}\sum_{cyc}bc(1+ab)=4+\frac{1}{2}\left(\sum_{cyc}ab-\sum_{cyc}ab^2c\right).$$

It remains to prove that

$$ab+bc+cd+da\ge ab^2c+bc^2d+cd^2a+da^2b.$$

Applying the familiar result $xy+yz+zt+tx\le\frac{1}{4}(x+y+z+t)^2$, we refer that

$$(ab+bc+cd+da)^2\ge 4(ab^2c+bc^2d+cd^2a+da^2b);$$

$$16=(a+b+c+d)^2\ge 4(ab+bc+cd+da).$$

Multiplying the above inequalities, we get the desired result. The equality holds for $a=b=c=d=1$ or $a=c=0$ (b,d arbitrary) or $b=d=0$ (a,c arbitrary).

∇

Example 1.2.9. *Let a,b,c be positive real numbers satisfying $a^2+b^2+c^2=3$. Prove that*

$$\frac{1}{a^3+2}+\frac{1}{b^3+2}+\frac{1}{c^3+2}\ge 1.$$

(Pham Kim Hung)

SOLUTION. According to **AM-GM**, we obtain

$$\sum_{cyc}\frac{1}{a^3+2}=\frac{3}{2}-\frac{1}{2}\sum_{cyc}\frac{a^3}{a^3+1+1}$$

$$\ge\frac{3}{2}-\frac{1}{2}\sum_{cyc}\frac{a^3}{3a}=1.$$

∇

Chapter 2

Cauchy-Schwarz and Hölder inequalities

2.1 Cauchy-Schwarz inequality and Applications

Theorem 2 (Cauchy-Schwarz inequality). *Let $(a_1, a_2, ..., a_n)$ and $(b_1, b_2, ..., b_n)$ be two sequences of real numbers. We have*

$$(a_1^2 + a_2^2 + ... + a_n^2)(b_1^2 + b_2^2 + ... + b_n^2) \geq (a_1b_1 + a_2b_2 + ... + a_nb_n)^2.$$

The equality holds if and only if $(a_1, a_2, ..., a_n)$ and $(b_1, b_2, ..., b_n)$ are proportional (there is a real number k for which $a_i = kb_i$ for all $i \in \{1, 2, ..., n\}$).

PROOF. I will give popular solutions to this theorem.

First solution. *(using quadratic form)* Consider the following function

$$f(x) = (a_1x - b_1)^2 + (a_2x - b_2)^2 + ... + (a_nx - b_n)^2$$

which is rewritten as

$$f(x) = (a_1^2 + a_2^2 + ... + a_n^2)x^2 - 2(a_1b_1 + a_2b_2 + ... + a_nb_n)x + (b_1^2 + b_2^2 + ... + b_n^2).$$

Since $f(x) \geq 0$ $\forall x \in \mathbb{R}$, we must have $\Delta_f \leq 0$ or

$$(a_1^2 + a_2^2 + ... + a_n^2)(b_1^2 + b_2^2 + ... + b_n^2) \geq (a_1b_1 + a_2b_2 + ... + a_nb_n)^2.$$

The equality holds if the equation $f(x) = 0$ has at least one root, or in other words $(a_1, a_2, ..., a_n)$ and $(b_1, b_2, ..., b_n)$ are proportional.

Second solution. *(using an identity)* The following identity is called **Cauchy-Schwarz** expansion. It helps prove **Cauchy-Schwarz** inequality immediately

$$(a_1^2+a_2^2+...+a_n^2)(b_1^2+b_2^2+...+b_n^2)-(a_1b_1+a_2b_2+...+a_nb_n)^2=\sum_{i,j=1}^{n}(a_ib_j-a_jb_i)^2.$$

Third solution. *(using* **AM-GM***)* This proof is used to prove **Hölder** inequality as well. Notice that, according to **AM-GM** inequality, we have

$$\begin{aligned}&\frac{a_i^2}{a_1^2+a_2^2+...+a_n^2}+\frac{b_i^2}{b_1^2+b_2^2+...+b_n^2}\\ &\geq\frac{2|a_ib_i|}{\sqrt{(a_1^2+a_2^2+...+a_n^2)(b_1^2+b_2^2+...+b_n^2)}}.\end{aligned}$$

Let i run from 1 to n and sum up all of these estimations. We get the conclusion.

∇

Which is basic inequality? The common answer is **AM-GM**. But what is the most *original* of the basic inequalities? I incline to answer **Cauchy-Schwarz** inequality. Why? Because **Cauchy-Schwartz** is so effective in proving symmetric inequalities, especially inequalities in three variables. It often provides pretty solutions as well. The following corollaries can contribute to the many applications it has.

Corollary 1. (Schwarz inequality). For any two sequences of real numbers $(a_1,a_2,...,a_n)$ and $(b_1,b_2,...,b_n)$, $(b_i>0\ \forall i\in\{1,2,...,n\})$, we have

$$\frac{a_1^2}{b_1}+\frac{a_2^2}{b_2}+...+\frac{a_n^2}{b_n}\geq\frac{(a_1+a_2+...+a_n)^2}{b_1+b_2+...+b_n}.$$

Solution. This result is directly obtained from **Cauchy-Schwarz**.

∇

Corollary 2. For every two sequences of real numbers $(a_1,a_2,...,a_n)$ and $(b_1,b_2,...,b_n)$, we always have

$$\sqrt{a_1^2+b_1^2}+\sqrt{a_2^2+b_2^2}+...+\sqrt{a_n^2+b_n^2}\geq\sqrt{(a_1+...+a_n)^2+(b_1+...+b_n)^2}.$$

Proof. By a simple induction, it suffices to prove the problem in the case $n=2$. In this case, the inequality becomes

$$\sqrt{a_1^2+b_1^2}+\sqrt{a_2^2+b_2^2}\geq\sqrt{(a_1+a_2)^2+(b_1+b_2)^2},$$

Squaring and reducing similar term, it simply becomes **Cauchy-Schwarz**

$$(a_1^2+a_2^2)(b_1^2+b_2^2) \geq (a_1b_1+a_2b_2)^2.$$

Certainly, the equality occurs iff $(a_1, a_2, ..., a_n)$ and $(b_1, b_2, ..., b_n)$ are proportional.

∇

Corollary 3. For any sequence of real numbers $a_1, a_2, ..., a_n$ we have

$$(a_1+a_2+...+a_n)^2 \leq n(a_1^2+a_2^2+...+a_n^2).$$

PROOF. Use **Cauchy-Schwarz** for the following sequences of n terms

$$(a_1, a_2, ..., a_n)\ ,\quad (1, 1, ..., 1).$$

∇

If applying **AM-GM** inequality is reduced *to gathering equal terms*, (in the analysis of the equality case) the **Cauchy-Schwarz** inequality is somewhat more flexible and generous. The following problems are essential and necessary because they include a lot of different ways of applying **Cauchy-Schwarz** accurately and effectively.

Example 2.1.1. *Let a, b, c be non-negative real numbers. Prove that*

$$\frac{a^2-bc}{2a^2+b^2+c^2}+\frac{b^2-ca}{2b^2+c^2+a^2}+\frac{c^2-ab}{2c^2+a^2+b^2} \geq 0.$$

(Pham Kim Hung)

SOLUTION. The inequality is equivalent to

$$\sum_{cyc}\frac{(a+b)^2}{a^2+b^2+2c^2} \leq 3.$$

According to **Cauchy-Schwarz** inequality, we have

$$\frac{(a+b)^2}{a^2+b^2+2c^2} \leq \frac{a^2}{a^2+c^2}+\frac{b^2}{b^2+c^2}.$$

That concludes

$$\sum_{cyc}\frac{(a+b)^2}{a^2+b^2+2c^2} \leq \sum_{cyc}\frac{a^2}{a^2+c^2}+\sum_{cyc}\frac{b^2}{b^2+c^2}=3.$$

Equality holds for $a=b=c$ and $a=b, c=0$ or its permutations.

∇

Example 2.1.2. *Suppose that $x, y, z \geq 1$ and $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=2$. Prove that*

$$\sqrt{x+y+z} \geq \sqrt{x-1}+\sqrt{y-1}+\sqrt{z-1}.$$

(Iran MO 1998)

SOLUTION. By hypothesis, we obtain

$$\frac{x-1}{x}+\frac{y-1}{y}+\frac{z-1}{z}=1.$$

According to **Cauchy-Schwarz**, we have

$$\sum_{cyc} x=\left(\sum_{cyc} x\right)\left(\sum_{cyc} \frac{x-1}{x}\right) \geq\left(\sum_{cyc} \sqrt{x-1}\right)^{2},$$

which implies

$$\sqrt{x+y+z} \geq \sqrt{x-1}+\sqrt{y-1}+\sqrt{z-1}.$$

∇

Example 2.1.3. *Let a, b, c be positive real numbers. Prove that*

$$\frac{a^{3}}{a^{3}+b^{3}+abc}+\frac{b^{3}}{b^{3}+c^{3}+abc}+\frac{c^{3}}{c^{3}+a^{3}+abc} \geq 1.$$

(Nguyen Van Thach)

SOLUTION. Let $x=\frac{b}{a}, y=\frac{c}{b}$ and $z=\frac{a}{c}$. Then we have

$$\frac{a^{3}}{a^{3}+b^{3}+abc}=\frac{1}{1+x^{3}+\frac{x}{z}}=\frac{1}{1+x^{3}+x^{2}z}=\frac{yz}{yz+x^{2}+xz}.$$

By **Cauchy-Schwarz** inequality, we deduce that

$$\sum_{cyc} \frac{yz}{yz+x^{2}+xz} \geq \frac{(xy+yz+zx)^{2}}{yz(yz+x^{2}+xy)+zx(zx+y^{2}+yx)+xy(xy+z^{2}+zy)}.$$

So it suffices to prove that

$$(xy+yz+zx)^{2} \geq \sum_{cyc} yz(yz+x^{2}+xz) \Leftrightarrow \sum_{cyc} x^{2}y^{2} \geq \sum_{cyc} x^{2}yz,$$

which is obvious. Equality holds for $x=y=z$ or $a=b=c$.

∇

Example 2.1.4. *Let a, b, c be three arbitrary real numbers. Denote*

$$x = \sqrt{b^2 - bc + c^2}, \; y = \sqrt{c^2 - ca + a^2}, \; z = \sqrt{a^2 - ab + b^2}.$$

Prove that

$$xy + yz + zx \geq a^2 + b^2 + c^2.$$

(Nguyen Anh Tuan, **VMEO 2006***)*

SOLUTION. Rewrite x, y in the following forms

$$x = \sqrt{\frac{3c^2}{4} + \left(b - \frac{c}{2}\right)^2}, \; y = \sqrt{\frac{3c^2}{4} + \left(a - \frac{c}{2}\right)^2}.$$

According to **Cauchy-Schwarz** inequality, we conclude

$$xy \geq \frac{3c^2}{4} + \frac{1}{4}(2b - c)(2a - c),$$

which implies

$$\sum_{cyc} xy \geq \frac{3}{4} \sum_{cyc} c^2 + \frac{1}{4} \sum_{cyc} (2b - c)(2a - c) = \sum_{cyc} a^2.$$

Comment. By the same approach, we can prove the following similar result

★ *Let a, b, c be three arbitrary real numbers. Denote*

$$x = \sqrt{b^2 + bc + c^2}, \; y = \sqrt{c^2 + ca + a^2}, \; z = \sqrt{a^2 + ab + b^2}.$$

Prove that

$$xy + yz + zx \geq (a + b + c)^2.$$

∇

Example 2.1.5. *Let a, b, c, d be non-negative real numbers. Prove that*

$$\frac{a}{b^2 + c^2 + d^2} + \frac{b}{a^2 + c^2 + d^2} + \frac{c}{a^2 + b^2 + d^2} + \frac{d}{a^2 + b^2 + c^2} \geq \frac{4}{a + b + c + d}.$$

(Pham Kim Hung)

SOLUTION. According to **Cauchy-Schwarz**, we have

$$\left(\frac{a}{b^2 + c^2 + d^2} + \frac{b}{a^2 + c^2 + d^2} + \frac{c}{a^2 + b^2 + d^2} + \frac{d}{a^2 + b^2 + c^2}\right)(a + b + c + d)$$

$$\geq \left(\sqrt{\frac{a^2}{b^2 + c^2 + d^2}} + \sqrt{\frac{b^2}{a^2 + c^2 + d^2}} + \sqrt{\frac{c^2}{a^2 + b^2 + d^2}} + \sqrt{\frac{d^2}{a^2 + b^2 + c^2}}\right)^2.$$

It remains to prove that

$$\sum_{cyc}\sqrt{\frac{a^2}{b^2+c^2+d^2}} \geq 2.$$

According to **AM-GM**

$$\sqrt{\frac{b^2+c^2+d^2}{a^2}} \leq \frac{1}{2}\left(\frac{b^2+c^2+d^2}{a^2}+1\right) = \frac{a^2+b^2+c^2+d^2}{2a^2}.$$

We can conclude that

$$\sum_{cyc}\sqrt{\frac{a^2}{b^2+c^2+d^2}} \geq \sum_{cyc}\frac{2a^2}{a^2+b^2+c^2+d^2} = 2$$

which is exactly the desired result. Equality holds if two of four numbers (a, b, c, d) are equal and the other ones are equal to 0 (for example, $(a, b, c, d) = (k, k, 0, 0)$).

Comment. Here is the general problem solvable by the same method.

★ *Let $a_1, a_2, ..., a_n$ be non-negative real numbers. Prove that*

$$\frac{a_1}{a_2^2+..+a_n^2}+\frac{a_2}{a_1^2+a_3^2+...+a_n^2}+...+\frac{a_n^2}{a_1^2+...+a_{n-1}^2} \geq \frac{4}{a_1+a_2+...+a_n}.$$

∇

Example 2.1.6. *Prove that for all positive real numbers a, b, c, d, e, f, we alwyas have*

$$\frac{a}{b+c}+\frac{b}{c+d}+\frac{c}{d+e}+\frac{d}{e+f}+\frac{e}{f+a}+\frac{f}{a+b} \geq 3.$$

(Nesbitt's inequality in six variables)

SOLUTION. According to **Cauchy-Schwarz** inequality

$$\sum_{cyc}\frac{a}{b+c} = \sum_{cyc}\frac{a^2}{ab+ac} \geq \frac{(a+b+c+d+e+f)^2}{ab+bc+cd+de+ef+fa+ac+ce+ea+bd+df+fb}.$$

Denote the denominator of the right fraction above by S. Certainly,

$$2S = (a+b+c+d+e+f)^2 - (a+d)^2 - (b+e) - (c+f)^2.$$

Applying **Cauchy-Schwarz** inequality again, we get

$$(1+1+1)\left[(a+d)^2+(b+e)^2+(c+f)^2\right] \geq (a+b+c+d+e+f)^2.$$

Thus $2S \leq \frac{2}{3}(a+b+c+d+e+f)^2$, which implies the desired result.

∇

Example 2.1.7. *Two real sequences* $(a_1, a_2, ..., a_n)$ *and* $(b_1, b_2, ..., b_n)$ *satisfy*

$$a_1^2 + a_2^2 + ... + a_n^2 = b_1^2 + b_2^2 + ... + b_n^2 = 1.$$

Prove the following inequality

$$(a_1b_2 - a_2b_1)^2 \le 2|a_1b_1 + a_2b_2 + ... + a_nb_n - 1|.$$

(Korea MO 2002)

SOLUTION. By **Cauchy-Schwarz,** the condition $\sum\limits_{i=1}^{n} a_i^2 = \sum\limits_{i=1}^{n} b_i^2 = 1$ yields

$$1 \ge a_1b_1 + a_2b_2 + ... + a_nb_n \ge -1.$$

According to the expansion of the **Cauchy-Schwarz** inequality, we have

$$(a_1^2 + ... + a_n^2)(b_1^2 + ... + b_n^2) - (a_1b_1 + ... + a_nb_n)^2 = \sum_{i,j=1}^{n} (a_ib_j - a_jb_i)^2 \ge (a_1b_2 - a_2b_1)^2$$

or equivalently

$$\left(1 - \sum_{i=1}^{n} a_ib_i\right)\left(1 + \sum_{i=1}^{n} a_ib_i\right) \ge (a_1b_2 - a_2b_1)^2.$$

That concludes

$$2|a_1b_1 + a_2b_2 + ... + a_nb_n - 1| \ge (a_1b_2 - a_2b_1)^2.$$

∇

Example 2.1.8. *Suppose* a, b, c *are positive real numbers with sum* 3. *Prove that*

$$\sqrt{a + \sqrt{b^2 + c^2}} + \sqrt{b + \sqrt{c^2 + a^2}} + \sqrt{c + \sqrt{a^2 + b^2}} \ge 3\sqrt{\sqrt{2} + 1}.$$

(Phan Hong Son)

SOLUTION. We rewrite the inequality in the following form (after squaring both sides)

$$\sum_{cyc} \sqrt{b^2 + c^2} + 2\sum_{cyc} \sqrt{\left(a + \sqrt{b^2 + c^2}\right)\left(b + \sqrt{c^2 + a^2}\right)} \ge 9\sqrt{2} + 6.$$

According to **Cauchy-Schwarz** inequality, we have

$$\sum_{cyc} \sqrt{\left(a + \sqrt{b^2 + c^2}\right)\left(b + \sqrt{c^2 + a^2}\right)} \ge \sum_{cyc} \sqrt{\left(a + \frac{b + c}{\sqrt{2}}\right)\left(b + \frac{c + a}{\sqrt{2}}\right)}$$

$$= \frac{1}{\sqrt{2}} \sum_{cyc} \sqrt{\left(\left(\sqrt{2}-1\right) a + 3\right) \left(\sqrt{2}-1)b + 3\right)}$$

$$\geq \frac{1}{\sqrt{2}} \sum_{cyc} \left(\left(\sqrt{2}-1\right) \sqrt{ab} + 3\right) = \left(1 - \frac{1}{\sqrt{2}}\right) \sum_{cyc} \sqrt{ab} + \frac{9}{\sqrt{2}}.$$

It remains to prove that

$$\sum_{cyc} \sqrt{a^2+b^2} + \left(2-\sqrt{2}\right) \sum_{cyc} \sqrt{ab} \geq 6.$$

This last inequality can be obtained directly from the following result for all $x, y \geq 0$

$$\sqrt{x^4+y^4} + \left(2-\sqrt{2}\right) xy \geq x^2 + y^2.$$

Indeed, the above inequality is equivalent to

$$x^4 + y^4 \geq \left(x^2 + y^2 - \left(2-\sqrt{2}\right) xy\right)^2 \Leftrightarrow 2(2-\sqrt{2})xy(x-y)^2 \geq 0$$

which is obviously true. Equality holds for $a = b = c = 1$.

∇

Example 2.1.9. *Suppose a, b, c are positive real numbers such that $abc = 1$. Prove the following inequality*

$$\frac{1}{a^2+a+1} + \frac{1}{b^2+b+1} + \frac{1}{c^2+c+1} \geq 1.$$

SOLUTION. By hypothesis, there exist three positive real numbers x, y, z for which

$$a = \frac{yz}{x^2},\ b = \frac{xz}{y^2},\ c = \frac{xy}{z^2}.$$

The inequality can be rewritten to

$$\sum_{cyc} \frac{x^4}{x^4 + x^2yz + y^2z^2} \geq 1.$$

According to **Cauchy-Schwarz**, we have

$$\text{LHS} \geq \frac{(x^2+y^2+z^2)^2}{x^4+y^4+z^4+xyz(x+y+z)+x^2y^2+y^2z^2+z^2x^2}.$$

It suffices to prove that

$$(x^2+y^2+z^2)^2 \geq x^4+y^4+z^4+xyz(x+y+z)+x^2y^2+y^2z^2+z^2x^2$$

which is equivalent

$$\sum_{cyc} x^2y^2 \geq xyz\sum_{cyc} x \Leftrightarrow \sum_{cyc} z^2(x-y)^2 \geq 0.$$

Equality holds for $x = y = z$, or $a = b = c = 1$.

∇

Example 2.1.10. *Let a, b, c be the side-lengths of a triangle. Prove that*

$$\frac{a}{3a-b+c} + \frac{b}{3b-c+a} + \frac{c}{3c-a+b} \geq 1.$$

(Samin Riasa)

SOLUTION. By **Cauchy-Schwarz,** we have

$$\begin{aligned} 4\sum_{cyc}\frac{a}{3a-b+c} &= \sum_{cyc}\frac{4a}{3a-b+c} \\ &= 3 + \sum_{cyc}\frac{a+b-c}{3a-b+c} \\ &\geq 3 + \frac{(a+b+c)^2}{\sum_{cyc}(a+b-c)(3a-b+c)} \\ &= 4. \end{aligned}$$

Equality holds for $a = b = c$.

∇

Example 2.1.11. *Let a, b, c be positive real numbers such that $a \leq b \leq c$ and $a + b + c = 3$. Prove that*

$$\sqrt{3a^2+1} + \sqrt{5a^2+3b^2+1} + \sqrt{7a^2+5b^2+3c^2+1} \leq 9.$$

(Pham Kim Hung)

SOLUTION. According to **Cauchy-Schwarz,** we have

$$\left(\sqrt{3a^2+1} + \sqrt{5a^2+3b^2+1} + \sqrt{7a^2+5b^2+3c^2+1}\right)^2 =$$

$$= \left(\frac{1}{\sqrt{6}}\cdot\sqrt{6(3a^2+1)} + \frac{1}{\sqrt{4}}\cdot\sqrt{4(5a^2+3b^2+1)} + \frac{1}{\sqrt{3}}\cdot\sqrt{3(7a^2+5b^2+3c^2+1)}\right)^2$$

$$\leq \left(\frac{1}{6}+\frac{1}{4}+\frac{1}{2}\right)\left[6(3a^2+1) + 4(5a^2+3b^2+1) + 3(7a^2+5b^2+3c^2+1)\right].$$

It remains to prove that

$$59a^2 + 27b^2 + 9c^2 \leq 95.$$

Notice that $a \leq b \leq c$, so we have

$$ab + bc + ca \geq 2ab + b^2 \geq 2a^2 + b^2$$

or

$$5a^2 + 3b^2 + c^2 \leq (a+b+c)^2 = 9 \Rightarrow 59a^2 + 27b^2 + 9c^2 \leq 95,$$

since $a \leq 1$. Equality holds for $a = b = c$.

∇

Example 2.1.12. *Let a, b, c be positive real numbers with sum 1. Prove that*

$$\frac{1+a}{1-a} + \frac{1+b}{1-b} + \frac{1+c}{1-c} \leq \frac{2a}{b} + \frac{2b}{c} + \frac{2c}{a}.$$

(Japan TST 2004)

SOLUTION. Rewrite this inequality in the form

$$\frac{3}{2} + \sum_{cyc} \frac{a}{b+c} \leq \sum_{cyc} \frac{a}{b} \Leftrightarrow \sum_{cyc} \left(\frac{a}{b} - \frac{a}{b+c} \right) \geq \frac{3}{2} \Leftrightarrow \sum_{cyc} \frac{ac}{b(b+c)} \geq \frac{3}{2}.$$

According to **Cauchy-Schwarz,**

$$\sum_{cyc} \frac{ac}{b(b+c)} = \sum_{cyc} \frac{a^2c^2}{abc(a+c)} \geq \frac{(ab+bc+ca)^2}{2abc(a+b+c)} \geq \frac{3}{2}.$$

Equality holds if and only if $a = b = c$.

∇

Example 2.1.13. (i). *Prove that for all non-negative real numbers x, y, z*

$$6(x+y-z)(x^2+y^2+z^2) + 27xyz \leq 10(x^2+y^2+z^2)^{3/2}.$$

(ii). *Prove that for all real numbers x, y, z then*

$$6(x+y+z)(x^2+y^2+z^2) \leq 27xyz + 10(x^2+y^2+z^2)^{3/2}.$$

(Tran Nam Dung)

SOLUTION. (*i*). For this part, it's necessary to be aware of the fact that the equality holds for $x = y = 2z$ up to permutation. This suggests using orientated estimations. Indeed, by **Cauchy-Schwarz**, we deduce that

$$\begin{aligned}
&10(x^2+y^2+z^2)^{3/2} - 6(x+y-z)(x^2+y^2+z^2)\\
&= (x^2+y^2+z^2)\left(10\sqrt{x^2+y^2+z^2} - 6(x+y-z)\right)\\
&= (x^2+y^2+z^2)\left(\frac{10}{3}\sqrt{(x^2+y^2+z^2)(2^2+2^2+1^2)} - 6(x+y-z)\right)\\
&\geq (x^2+y^2+z^2)\left(\frac{10(2x+2y+z)}{3} - 6(x+y-z)\right)\\
&= \frac{10(x^2+y^2+z^2)(2x+2y+28z)}{3}.
\end{aligned}$$

Then, according to the weighted **AM-GM** inequality, we deduce that

$$x^2+y^2+z^2 = 4\cdot\frac{x^2}{4} + 4\cdot\frac{y^2}{4} + z^2 \geq 9\sqrt[9]{\frac{x^8y^8z^2}{4^8}}$$

$$2x+2y+28z = 2x+2y+7\cdot 4z \geq 9\sqrt[9]{(2x)(2y)(4z)^7} \geq 9\sqrt[9]{4^8xyz^7}.$$

Therefore

$$10(x^2+y^2+z^2)^{3/2} - 6(x+y-z)(x^2+y^2+z^2) \geq 27xyz.$$

(*ii*). The problem is obvious if $x = y = z = 0$. Otherwise, we may assume that $x^2+y^2+z^2 = 9$ without loss of generality (if you don't know how to normalize an inequality yet, yake a peak at page 120). The problem becomes

$$2(x+y+z) \leq xyz + 10.$$

Suppose $|x| \leq |y| \leq |z|$. According to **Cauchy-Schwarz**, we get

$$\begin{aligned}
\left[2(x+y+z) - xyz\right]^2 &= \left[(2(x+y) + (2-xy)z\right]^2\\
&\leq \left((x+y)^2+z^2\right)\left(2^2+(2-xy)^2\right)\\
&= (9+2xy)(8-4xy+x^2y^2)\\
&= 72 - 20xy + x^2y^2 + 2x^3y^3\\
&= 100 + (xy+2)^2(xy-7).
\end{aligned}$$

Because $|x| \leq |y| \leq |z|$, $z^2 \geq 3 \Rightarrow 2xy \leq x^2+y^2 \leq 6 \Rightarrow xy - 7 < 0$. This yields

$$(2(x+y+z) - xyz)^2 \leq 100 \Rightarrow 2(x+y+z) \leq 10 + xyz.$$

Equality holds for $(x, y, z) = (-k, 2k, 2k)$ $(\forall k \in \mathbb{R})$ up to permutation.

∇

Example 2.1.14. *Let a, b, c, d be four positive real numbers such that $r^4 = abcd \geq 1$. Prove the following inequality*

$$\frac{ab+1}{a+1}+\frac{bc+1}{b+1}+\frac{cd+1}{c+1}+\frac{da+1}{d+1}\geq\frac{4(1+r^2)}{1+r}.$$

(Vasile Cirtoaje, **Crux**)

SOLUTION. The hypothesis implies the existence of four positive real numbers x, y, z, t such that

$$a=\frac{ry}{x}\ ,\quad b=\frac{rz}{y}\ ,\quad c=\frac{rt}{z}\ ,\quad d=\frac{rx}{t}.$$

The inequality is therefore rewritten in the following form

$$\sum_{cyc}\frac{\frac{r^2z}{x}+1}{\frac{ry}{x}+1}\geq\frac{4(r^2+1)}{r+1}\ \Leftrightarrow\ \sum_{cyc}\frac{r^2z+x}{ry+x}\geq\frac{4(r^2+1)}{r+1}.$$

We need to prove that $A+(r^2-1)B\geq\dfrac{4(r^2+1)}{r+1}$, where

$$A=\sum_{cyc}\frac{x+z}{ry+x}\ ;\ B=\sum_{cyc}\frac{z}{ry+x}.$$

By **AM-GM**, we have of course

$$\begin{aligned}4r\sum_{cyc}xy+8(xz+yt) &= \Big[4(r-1)(x+z)(y+t)\Big]+4\Big[(x+z)(y+t)+2(xz+yt)\Big]\\ &\leq (r-1)\Big(\sum_{cyc}x\Big)^2+2\Big(\sum_{cyc}x\Big)^2=(r+1)\Big(\sum_{cyc}x\Big)^2\end{aligned}$$

According to **Cauchy-Schwarz**, with the notice that $r\geq 1$, we conclude

$$\begin{aligned}A= &\ (x+z)\left(\frac{1}{ry+x}+\frac{1}{rt+z}\right)+(y+t)\left(\frac{1}{rx+y}+\frac{1}{rz+t}\right)\\ \geq &\ \frac{4(x+z)}{x+z+ry+rt}+\frac{4(y+t)}{y+t+rx+rz}\\ \geq &\ \frac{4(x+y+z+t)^2}{(x+z)^2+(y+t)^2+2r(x+z)(y+t)}\geq\frac{8}{r+1},\\ B\geq &\ \frac{(x+y+z+t)^2}{z(ry+x)+t(rz+y)+x(rt+z)+y(rx+t)}\\ \geq &\ \frac{(x+y+z+t)^2}{r(xy+yz+zt+tx)+2(xz+yt)}\geq\frac{4}{r+1},\end{aligned}$$

and the conclusion follows. Equality holds for $a=b=c=d=r$.

∇

Example 2.1.15. *Let a, b, c be non-negative real numbers. Prove that*

$$\frac{a^2}{a^2+2(a+b)^2}+\frac{b^2}{b^2+2(b+c)^2}+\frac{c^2}{c^2+2(c+a)^2}\ge\frac{1}{3}.$$

(Pham Kim Hung)

SOLUTION. We denote $x=\frac{b}{a}, y=\frac{c}{b}, z=\frac{a}{c}$. The problem becomes

$$\sum_{cyc}\frac{1}{1+2(x+1)^2}\ge\frac{1}{3}.$$

Because $xyz=1$, there exist three positive real numbers m, n, p such that

$$x=\frac{np}{m^2}, y=\frac{mp}{n^2}, z=\frac{mn}{p^2}.$$

It remains to prove that

$$\sum_{cyc}\frac{m^4}{m^4+2(m^2+np)^2}\ge\frac{1}{3}.$$

According to **Cauchy-Schwarz,** we deduce that

$$\text{LHS}\ge\frac{(m^2+n^2+p^2)^2}{m^4+n^4+p^4+2(m^2+np)^2+2(n^2+mp)^2+2(p^2+mn)^2}.$$

Since we have

$$3\left(\sum_{cyc}m^2\right)^2-\sum_{cyc}m^4-2\sum_{cyc}(m^2+np)^2=\sum_{cyc}m^2(n-p)^2\ge 0,$$

the proof is finished. Equality holds for $a=b=c$.

∇

Example 2.1.16. *Let a, b, c be non-negative real numbers. Prove that*

$$\frac{a}{b^2+c^2}+\frac{b}{a^2+c^2}+\frac{c}{a^2+b^2}\ge\frac{4}{5}\left(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right).$$

(Pham Kim Hung)

SOLUTION. Applying **Cauchy-Schwarz,** we obtain

$$\left(\sum_{cyc}\frac{a}{b^2+c^2}\right)\left(\sum_{cyc}a(b^2+c^2)\right)\ge(a+b+c)^2.$$

It remains to prove that

$$\frac{(a+b+c)^2}{ab(a+b)+bc(b+c)+ca(c+a)}\ge\frac{4}{5}\cdot\frac{a^2+b^2+c^2+3(ab+bc+ca)}{ab(a+b)+bc(b+c)+ca(c+a)+2abc}.$$

Let $S = \sum_{\text{cyc}} a^2$, $P = \sum_{\text{cyc}} ab$ and $Q = \sum_{\text{cyc}} ab(a+b)$. The above inequality becomes

$$\frac{5(S+2P)}{Q} \geq \frac{4(S+3P)}{Q+2abc} \Leftrightarrow SQ + 10abcS + 20abcP \geq 2PQ.$$

Clearly, we have

$$PQ = \sum_{sym} a^2b^2(a+b) + 2abc(S+P),$$

$$SQ \geq \sum_{sym} ab(a+b)(a^2+b^2) \geq 2\sum_{sym} a^2b^2(a+b).$$

This finishes the prof. Equality holds for $a = b, c = 0$ up to permutations.

∇

Like with the **AM-GM** inequality, there is no fixed way of applying the **Cauchy-Schwarz** inequality. It depends on the kind of problem and on how flexible you are in using this inequality. A consistent application of **Cauchy-Schwarz,** the **Hölder** inequality is, in fact, a typical extension. Although **Hölder** is somehow neglected in the world of inequalities, almost disregarded in comparison with **AM-GM** or **Cauchy-Schwarz,** this book will emphasize this inequality's importance. I place **Hölder** inequality in a subsection of the **Cauchy-Schwarz** inequality because it is a natural generalization of **cauchy-Schwarz** and its application is not so different than **Cauchy-Schwarz** inequality's application.

2.2 Hölder Inequality

Theorem 3 (Hölder inequality). *For m sequences of positive real numbers $(a_{1,1}, a_{1,2}, \ldots, a_{1,n})$, $(a_{2,1}, a_{2,2}, \ldots, a_{2,n}), \ldots, (a_{m,1}, a_{m,2}, \ldots, a_{m,n})$, we have*

$$\prod_{i=1}^{m}\left(\sum_{j=1}^{n} a_{i,j}\right) \geq \left(\sum_{j=1}^{n} \sqrt[m]{\prod_{i=1}^{m} a_{i,j}}\right)^m.$$

Equality occurs if and only if these m sequences are pairwise proportional. Cauchy-Schwarz inequality is a direct corollary of Hölder inequality for $m = 2$.

Corollary 1. *Let $a, b, c, x, y, z, t, u, v$ be positive real numbers. We always have*

$$(a^3+b^3+c^3)(x^3+y^3+z^3)(t^3+u^3+v^3) \geq (axt+byu+czv)^3.$$

PROOF. This is a direct corollary of **Hölder** inequality for $m = n = 3$. I choose this particular case of **Hölder** for a detailed proof because it exemplifies the proof of the general **Hölder** inequality.

According to **AM-GM**, we deduce that

$$3 = \sum_{cyc} \frac{a^3}{a^3+b^3+c^3} + \sum_{cyc} \frac{x^3}{x^3+y^3+z^3} + \sum_{cyc} \frac{m^3}{m^3+m^3+p^3}$$
$$\geq \sum_{cyc} \frac{3axm}{\sqrt[3]{(a^3+b^3+c^3)(x^3+y^3+z^3)(m^3+n^3+p^3)}}.$$

That means

$$axm + byn + czp \leq \sqrt[3]{(a^3+b^3+c^3)(x^3+y^3+z^3)(m^3+n^3+p^3)}.$$

∇

Corollary 2. *Let $a_1, a_2, ..., a_n$ be positive real numbers. Prove that*

$$(1+a_1)(1+a_2)...(1+a_n) \geq (1+\sqrt[n]{a_1a_2...a_n})^n.$$

PROOF. Applying **AM-GM**, we have

$$\frac{1}{1+a_1} + \frac{1}{1+a_2} + ... + \frac{1}{1+a_n} \geq \frac{n}{\sqrt[n]{(1+a_1)(1+a_2)...(1+a_n)}},$$
$$\frac{a_1}{1+a_1} + \frac{a_2}{1+a_2} + ... + \frac{a_n}{1+a_n} \geq \frac{n\sqrt[n]{a_1a_2...a_n}}{\sqrt[n]{(1+a_1)(1+a_2)...(1+a_n)}}.$$

These two inequalities, added up, give the desired result.

∇

Why do we sometimes neglect **Hölder** inequality? Despite its strong application, its sophisticated expression (with m sequences, each of which has n terms), makes it confusing the first time we try using it. If you are still hesitating, the book will try to convince you that **Hölder** is really effective and easy to use. Don't be afraid of familiarize yourself with it! Generally, a lot of difficult problems turn into very simple ones just after a deft application of **Hölder** inequality.

•

Example 2.2.1. *Let a, b, c be positive real numbers. Prove that*

$$\frac{a}{\sqrt{a^2+8bc}} + \frac{b}{\sqrt{b^2+8ac}} + \frac{c}{\sqrt{c^2+8ab}} \geq 1.$$

(IMO 2001, A2)

SOLUTION. Applying **Hölder** inequality for three sequences, each of which has three terms (actually, that's corollary 1), we deduce that

$$\left(\sum_{cyc}\frac{a}{\sqrt{a^2+8bc}}\right)\left(\sum_{cyc}\frac{a}{\sqrt{a^2+8bc}}\right)\left(\sum_{cyc}a(a^2+8bc)\right)\ge(a+b+c)^3.$$

So it suffices to prove that

$$(a+b+c)^3\ge\sum_{cyc}a(a^2+8bc)$$

or equivalently

$$c(a-b)^2+a(b-c)^2+b(c-a)^2\ge 0.$$

∇

Example 2.2.2. *Let a,b,c be positive real numbers such that $abc=1$. Prove that*

$$\frac{a}{\sqrt{7+b+c}}+\frac{b}{\sqrt{7+c+a}}+\frac{c}{\sqrt{7+a+b}}\ge 1,$$
$$\frac{a}{\sqrt{7+b^2+c^2}}+\frac{b}{\sqrt{7+c^2+a^2}}+\frac{c}{\sqrt{7+a^2+b^2}}\ge 1.$$

With the same condition, determine if the following inequality is true or false.

$$\frac{a}{\sqrt{7+b^3+c^3}}+\frac{b}{\sqrt{7+c^3+a^3}}+\frac{c}{\sqrt{7+a^3+b^3}}\ge 1.$$

(Pham Kim Hung)

SOLUTION. For the first one, apply **Hölder** inequality in the following form

$$\left(\sum_{cyc}\frac{a}{\sqrt{7+b+c}}\right)\left(\sum_{cyc}\frac{a}{\sqrt{7+b+c}}\right)\left(\sum_{cyc}a(7+b+c)\right)\ge(a+b+c)^3$$

It's enough to prove that

$$(a+b+c)^3\ge 7(a+b+c)+2(ab+bc+ca).$$

Because $a+b+c\ge 3\sqrt[3]{abc}=3$,

$$(a+b+c)^3\ge 7(a+b+c)+\frac{2}{3}(a+b+c)^2\ge 7(a+b+c)+2(ab+bc+ca).$$

For the second one, apply **Hölder** inequality in the following form

$$\left(\sum_{cyc}\frac{a}{\sqrt{7+b^2+c^2}}\right)\left(\sum_{cyc}\frac{a}{\sqrt{7+b^2+c^2}}\right)\left(\sum_{cyc}a(7+b^2+c^2)\right)\ge(a+b+c)^3.$$

On the other hand

$$\sum_{cyc} a(7+b^2+c^2) = 7(a+b+c)+(a+b+c)(ab+bc+ca)-3abc$$

$$\le 7(a+b+c)+\frac{1}{3}(a+b+c)^3-3 \le (a+b+c)^3.$$

Equality holds for $a=b=c=1$ for both parts.

The third one is not true. Indeed, we only need to choose $a \to 0$ and $b=c \to +\infty$, or namely, $a=10^{-4}, b=c=100$.

∇

Example 2.2.3. *Let a,b,c be positive real numbers. Prove that for all natural numbers k, $(k \ge 1)$, the following inequality holds*

$$\frac{a^{k+1}}{b^k}+\frac{b^{k+1}}{c^k}+\frac{c^{k+1}}{a^k} \ge \frac{a^k}{b^{k-1}}+\frac{b^k}{c^{k-1}}+\frac{c^k}{a^{k-1}}.$$

SOLUTION. According to **Hölder** inequality, we deduce that

$$\left(\frac{a^{k+1}}{b^k}+\frac{b^{k+1}}{c^k}+\frac{c^{k+1}}{a^k}\right)^{k-1}(a+b+c) \ge \left(\frac{a^k}{b^{k-1}}+\frac{b^k}{c^{k-1}}+\frac{c^k}{a^{k-1}}\right)^k.$$

It suffices to prove that

$$\frac{a^k}{b^{k-1}}+\frac{b^k}{c^{k-1}}+\frac{c^k}{a^{k-1}} \ge a+b+c,$$

which follows from **Hölder** inequality

$$\left(\frac{a^k}{b^{k-1}}+\frac{b^k}{c^{k-1}}+\frac{c^k}{a^{k-1}}\right)(b+c+a)^{k-1} \ge (a+b+c)^k.$$

Equality holds for $a=b=c$. Notice that this problem is still true for every rational number k $(k \ge 1)$, and therefore it's still true for every real number k $(k \ge 1)$.

∇

Example 2.2.4. *Let a,b,c be positive real numbers. Prove that*

$$(a^5-a^2+3)(b^5-b^2+3)(c^5-c^2+3) \ge (a+b+c)^3.$$

(Titu Andreescu, **USA MO 2002**)

SOLUTION. According to **Hölder** inequality, we conclude that

$$\prod_{cyc}(a^5-a^2+3) = \prod_{cyc}\left(a^3+2+(a^3-1)(a^2-1)\right) \ge \prod(a^3+2)$$

$$=(a^3+1+1)(1+b^3+1)(1+1+c^3) \ge (a+b+c)^3.$$

∇

Example 2.2.5. *Suppose a, b, c are three positive real numbers verifying $ab+bc+ca = 3$. Prove that*

$$(1+a^2)(1+b^2)(1+c^2) \geq 8.$$

(Michael Rozenberg)

SOLUTION. The inequality is directly obtained from **Hölder** inequality

$$(a^2b^2+a^2+b^2+1)(b^2+c^2+b^2c^2+1)(a^2+a^2c^2+c^2+1) \geq (1+ab+bc+ca)^4.$$

∇

Example 2.2.6. *Let a, b, c be positive real numbers which sum up to 1. Prove that*

$$\frac{a}{\sqrt[3]{a+2b}} + \frac{b}{\sqrt[3]{b+2c}} + \frac{c}{\sqrt[3]{c+2a}} \geq 1.$$

(Pham Kim Hung)

SOLUTION. This inequality is directly obtained from by **Hölder** inequality

$$\left(\sum_{cyc} \frac{a}{\sqrt[3]{a+2b}}\right)\left(\sum_{cyc} \frac{a}{\sqrt[3]{a+2b}}\right)\left(\sum_{cyc} \frac{a}{\sqrt[3]{a+2b}}\right)\left(\sum_{cyc} a(a+2b)\right) \geq \left(\sum a\right)^4 = 1$$

because

$$\sum_{cyc} a(a+2b) = (a+b+c)^2 = 1.$$

∇

Example 2.2.7. *Let a, b, c be positive real numbers. Prove that*

$$a^2(b+c)+b^2(c+a)+c^2(a+b) \geq (ab+bc+ca)\sqrt[3]{(a+b)(b+c)(c+a)}.$$

(Pham Kim Hung)

SOLUTION. Notice that the following expressions are equal to each other

$$a^2(b+c)+b^2(c+a)+c^2(a+b),$$
$$b^2(c+a)+c^2(a+b)+a^2(b+c),$$
$$ab(a+b)+bc(b+c)+ca(c+a).$$

According to **Hölder** inequality, we get that

$$\left(\sum_{cyc} a^2(b+c)\right)^3 \geq \left(\sum_{cyc} ab\sqrt[3]{(a+b)(b+c)(c+a)}\right)^3$$

which is exactly the desired result. Equality holds for $a = b = c$.

∇

Example 2.2.8. *Let a, b, c, d be positive real numbers such that $abcd = 1$. Prove that*

$$4^4(a^4+1)(b^4+1)(c^4+1)(d^4+1) \geq \left(a+b+c+d+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)^4.$$

(Gabriel Dospinescu)

SOLUTION. By **Hölder** inequality, we get that

$$(a^4+1)(1+b^4)(1+c^4)(1+d^4) \geq (a+bcd)^4 = \left(a+\frac{1}{a}\right)^4$$

$$\Rightarrow \sqrt[4]{(a^4+1)(b^4+1)(c^4+1)(d^4+1)} \geq a+\frac{1}{a}$$

$$\Rightarrow 4\sqrt[4]{(a^4+1)(b^4+1)(c^4+1)(d^4+1)} \geq \sum_{cyc} a + \sum_{cyc} \frac{1}{a}.$$

Equality holds for $a = b = c = d = 1$.

∇

Example 2.2.9. *Let a, b, c be positive real numbers. Prove that*

$$(a^2+ab+b^2)(b^2+bc+c^2)(c^2+ca+a^2) \geq (ab+bc+ca)^3.$$

SOLUTION. Applying **Hölder** inequality, we obtain

$$(a^2+ab+b^2)(b^2+bc+c^2)(c^2+ca+a^2)$$

$$= (ab+a^2+b^2)(a^2+ac+c^2)(b^2+c^2+bc) \geq (ab+ac+bc)^3.$$

∇

If an inequality can be solved by **Hölder** inequality, it can be solved by **AM-GM** inequality, too. Why? Because the proof of **Hölder** inequality only uses **AM-GM**. For instance, in example 2.2.1, we can use **AM-GM** directly in the following way

Let $M = a+b+c$. According to **AM-GM**, we have

$$\frac{a}{\sqrt{b^2+8ac}} + \frac{a}{\sqrt{b^2+8ac}} + \frac{a(b^2+8ac)}{(a+b+c)^3} \geq \frac{3a}{a+b+c}.$$

Our work on the LHS is to build up two other similar inequalities then sum up all of them.

But what is the difference between **AM-GM** and **Cauchy-Schwarz** and **Hölder**? Although both **Cauchy-Schwarz** inequality and **Hölder** inequality can be proved by **AM-GM**, they have a great advantage in application. They make a long and complicate solution through **AM-GM**, shorter and more intuitive. Let's see the following example to clarify this advantage.

Example 2.2.10. *Suppose that a, b, c are positive real numbers satisfying the condition $3\max(a^2, b^2, c^2) \le 2(a^2+b^2+c^2)$. Prove that*

$$\frac{a}{\sqrt{2b^2+2c^2-a^2}} + \frac{b}{\sqrt{2c^2+2a^2-b^2}} + \frac{c}{\sqrt{2a^2+2b^2-c^2}} \ge \sqrt{3}.$$

SOLUTION. By **Hölder**, we deduce that

$$\left(\sum_{cyc} \frac{a}{\sqrt{2b^2+2c^2-a^2}}\right)\left(\sum_{cyc} \frac{a}{\sqrt{2b^2+2c^2-a^2}}\right)\left(\sum_{cyc} a(2b^2+2c^2-a^2)\right) \ge (a+b+c)^3.$$

It remains to prove that

$$(a+b+c)^3 \ge 3\sum_{cyc} a(2b^2+2c^2-a^2).$$

Rewrite this one in the following form

$$3\left(abc - \prod_{cyc}(a-b+c)\right) + 2\left(\sum_{cyc} a^3 - 3abc\right) \ge 0,$$

which is obvious (for a quick proof that the first term is bigger than 0, replace $a-b+c = x$, etc). Equality holds for $a=b=c$.

∇

How can this problem be solved by **AM-GM**? Of course, it is a bit more difficult. Let's see that

$$\sum_{cyc} \frac{a}{\sqrt{2b^2+2c^2-a^2}} + \sum_{cyc} \frac{a}{\sqrt{2b^2+2c^2-a^2}} + \sum_{cyc} \frac{3\sqrt{3}a(2b^2+2c^2-a^2)}{(a+b+c)^3} \ge$$

$$\ge 3\sum_{cyc} \frac{\sqrt{3}a}{a+b+c} = 3\sqrt{3}.$$

To use **AM-GM** now, we must be aware of multiplying $3\sqrt{3}$ to the fraction

$$\frac{a(2b^2+2c^2-a^2)}{(a+b+c)^3}$$

in order to have

$$\frac{a}{\sqrt{2b^2+2c^2-a^2}} = \frac{a}{\sqrt{2b^2+2c^2-a^2}} = \frac{3\sqrt{3}a(2b^2+2c^2-a^2)}{(a+b+c)^3}$$

in case $a=b=c$. Why are **Hölder** and **Cauchy-Schwarz** more advantageous? Because, in stead of being conditioned by an "equal property" like **AM-GM** is, **Hölder** and **Cauchy-Schwarz** are conditioned by "proportional property". This feature makes **Hölder** and **Cauchy-Schwarz** easier to use in a lot of situations. Furthermore, **Hölder** is very effective in proving problems which involve roots (helps us get rid of the square root easily, for example).

Chapter 3

Chebyshev Inequality

3.1 Chebyshev Inequality and Applications

Theorem 4 (Chebyshev inequality). *Suppose $(a_1, a_2, ..., a_n)$ and $(b_1, b_2, ..., b_n)$ are two increasing sequences of real numbers, then*

$$a_1b_1 + a_2b_2 + ... + a_nb_n \geq \frac{1}{n}(a_1 + a_2 + ... + a_n)(b_1 + b_2 + ... + b_n).$$

PROOF. By directly expanding, we have

$$n(a_1b_1 + a_2b_2 + ... + a_nb_n) - (a_1 + a_2 + ... + a_n)(b_1 + b_2 + ... + b_n) =$$
$$= \sum_{i,j=1}^{n} (a_i - a_j)(b_i - b_j) \geq 0.$$

Comment. By the same proof, we also conclude that if the sequence $(a_1, a_2, ..., a_n)$ is increasing but the sequence $(b_1, b_2, ..., b_n)$ is decreasing, then

$$a_1b_1 + a_2b_2 + ... + a_nb_n \leq \frac{1}{n}(a_1 + a_2 + ... + a_n)(b_1 + b_2 + ... + b_n).$$

∇

For symmetric problems, we can rearrange the order of variables so that the condition of Chebyshev inequality is satisfied. Generally, solutions by Chebyshev inequality are more concise those that by other basic inequalities. Let's consider the following simple example

Example 3.1.1. *Let $a_1, a_2, ..., a_n$ be positive real numbers with sum n. Prove that*

$$a_1^{n+1} + a_2^{n+1} + ... + a_n^{n+1} \geq a_1^n + a_2^n + ... + a_n^n.$$

SOLUTION. To solve this problem by **AM-GM**, we must go through two steps: first, prove $n\sum_{i=1}^{n} a_i^{n+1} + n \geq (n+1)\sum_{i=1}^{n} a_i^n$, and then prove $\sum_{i=1}^{n} a_i^n \geq n$. To solve it by **Cauchy-Schwarz**, we must use an inductive approach eventually. However, this problem follows from **Chebyshev** inequality immediately with the notice that the sequences $(a_1, a_2, ..., a_n)$ and $(a_1^n, a_2^n, ..., a_n^n)$ can be rearranged so that they are increasing at once.

∇

Now we continue with some applications of **Chebyshev** inequality.

Example 3.1.2. *Let a, b, c, d be positive real numbers such that $a^2+b^2+c^2+d^2 = 4$. Prove the following inequality*

$$\frac{a^2}{b+c+d} + \frac{b^2}{c+d+a} + \frac{c^2}{d+a+b} + \frac{d^2}{a+b+c} \geq \frac{4}{3}.$$

SOLUTION. Notice that if (a, b, c, d) is arranged in an increasing order then

$$\frac{1}{b+c+d} \geq \frac{1}{c+d+a} \geq \frac{1}{d+a+b} \geq \frac{1}{a+b+c}.$$

Therefore, by **Chebyshev** inequality, we have

$$\begin{aligned} 4\text{LHS} &\geq \left(\sum_{cyc} a^2\right)\left(\sum_{cyc} \frac{1}{b+c+d}\right) \\ &\geq \frac{16(a^2+b^2+c^2+d^2)}{3(a+b+c+d)} \\ &\geq \frac{4\sqrt{4(a^2+b^2+c^2+d^2)}}{3}. \end{aligned}$$

That implies

$$\frac{a^2}{b+c+d} + \frac{b^2}{c+d+a} + \frac{c^2}{d+a+b} + \frac{d^2}{a+b+c} \geq \frac{4}{3}.$$

∇

Example 3.1.3. *Suppose that the real numbers $a, b, c > 1$ satisfy the condition*

$$\frac{1}{a^2-1} + \frac{1}{b^2-1} + \frac{1}{c^2-1} = 1.$$

Prove that

$$\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} \leq 1.$$

(Poru Loh, Crux)

SOLUTION. Notice that if $a \geq b \geq c$ then we have

$$\frac{a-2}{a+1} \geq \frac{b-2}{b+1} \geq \frac{c-2}{c+1} \quad ; \quad \frac{a+2}{a-1} \leq \frac{b+2}{b-1} \leq \frac{c+2}{c-1}.$$

Chebyshev inequality affirms that

$$3\left(\sum_{cyc} \frac{a^2-4}{a^2-1}\right) \leq \left(\sum_{cyc} \frac{a-2}{a+1}\right)\left(\sum_{cyc} \frac{a+2}{a-1}\right).$$

By hypothesis, the left-hand expression is equal to 0, which means

$$\frac{a-2}{a+1} + \frac{b-2}{b+1} + \frac{c-2}{c+1} \geq 0$$

which is equivalent to the desired result. Equality holds for $a = b = c = 2$.

∇

Example 3.1.4. *Let a, b, c, d, e be non-negative real numbers such that*

$$\frac{1}{4+a} + \frac{1}{4+b} + \frac{1}{4+c} + \frac{1}{4+d} + \frac{1}{4+e} = 1.$$

Prove that

$$\frac{a}{4+a^2} + \frac{b}{4+b^2} + \frac{c}{4+c^2} + \frac{d}{4+d^2} + \frac{e}{4+e^2} \leq 1.$$

SOLUTION. The hypothesis implies that $\sum_{cyc} \frac{1-a}{4+a} = 0$. We need to prove that

$$\sum_{cyc} \frac{1}{4+a} \geq \sum_{cyc} \frac{a}{4+a^2} \Leftrightarrow \sum_{cyc} \frac{1-a}{4+a} \cdot \frac{1}{4+a^2} \geq 0.$$

Assume that $a \geq b \geq c \geq d \geq e$, then

$$\frac{1-a}{4+a} \leq \frac{1-b}{4+b} \leq \frac{1-c}{4+c} \leq \frac{1-d}{4+d} \leq \frac{1-e}{4+e} \, ;$$
$$\frac{1}{4+a^2} \leq \frac{1}{4+b^2} \leq \frac{1}{4+c^2} \leq \frac{1}{4+d^2} \leq \frac{1}{4+e^2} \, ;$$

Applying **Chebyshev** inequality for the monotone sequences above, we get the desired result. Equality holds for $a = b = c = d = e = 1$.

∇

Example 3.1.5. *Suppose that a, b, c, d are four positive real numbers satisfying $a + b + c + d = 4$. Prove that*

$$\frac{1}{11+a^2} + \frac{1}{11+b^2} + \frac{1}{11+c^2} + \frac{1}{11+d^2} \leq \frac{1}{3}.$$

(Pham Kim Hung)

SOLUTION. Rewrite the inequality in the following form

$$\sum_{cyc}\left(\frac{1}{11+a^2}-\frac{1}{12}\right)\ge 0$$

or equivalently

$$\sum_{cyc}(1-a)\cdot\frac{a+1}{a^2+11}\ge 0.$$

Notice that if (a,b,c,d) is arranged in an increasing order then

$$\frac{a+1}{a^2+11}\ge\frac{b+1}{b^2+11}\ge\frac{c+1}{c^2+11}\ge\frac{d+1}{d^2+11}.$$

The desired results follows immediately from the **Chebyshev** inequality.

∇

Example 3.1.6. *Let a,b,c be three positive real numbers with sum 3. Prove that*

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\ge a^2+b^2+c^2.$$

(Vasile Cirtoaje, Romania TST 2006)

SOLUTION. Rewrite the inequality in the form

$$\sum_{cyc}a^2b^2\ge a^2b^2c^2\sum_{cyc}a^2 \Leftrightarrow \sum_{cyc}a^2b^2(1+c+c^2+c^3)(1-c)\ge 0.$$

Notice that if $ab\le 2$ and $a\ge b$ then

$$a^2(1+b+b^2+b^2)\ge b^2(1+a+a^2+a^3).$$

Indeed, this one is equivalent to $(a+b+ab-a^2b^2)(a-b)\ge 0$, which is obviously true because $ab\le 2$. From this property, we conclude that if all ab,bc,ca are smaller than 2 then **Chebyshev** inequality yields

$$\sum_{cyc}a^2b^2(1+c+c^2+c^2)(1-c)\ge\left(\sum_{sym}a^2b^2(1+c+c^2+c^3)\right)\left(\sum_{sym}(1-c)\right)=0.$$

Otherwise, suppose $ab\ge 2$. Clearly, $a+b\ge 2\sqrt{2}$, so $c\le 3-2\sqrt{2}$ and $c^2<\frac{1}{9}$. That means

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}>9>a^2+b^2+c^2.$$

The proof is finished. Equality holds for $a = b = c = 1$.

∇

In the following pages, we will discuss a special method of applying **Chebyshev** that is very effective and widely used. This technique is generally called "**Chebyshev** associate technique".

3.2 The Chebyshev Associate Technique

Let's analyze the following inequality

Example 3.2.1. *Suppose a, b, c, d are positive real numbers such that*

$$a + b + c + d = a^{-1} + b^{-1} + c^{-1} + d^{-1}.$$

Prove the inequality

$$2(a + b + c + d) \geq \sqrt{a^2 + 3} + \sqrt{b^2 + 3} + \sqrt{c^2 + 3} + \sqrt{d^2 + 3}.$$

(Pham Kim Hung)

SOLUTION. A cursory look at this inequality will leave you hesitating. The relationship between the variables a, b, c, d appears to be obscure and very hard to transform; moreover, the problem involves square roots. How can use handle this situation? Surprisingly enough, a simple way of applying **Chebyshev** can draw the enigmatic curtain. Let's discover the method!

By hypothesis, we have

$$\sum_{cyc} \frac{1}{a} = \sum_{cyc} a \Leftrightarrow \sum_{cyc} \left(a - \frac{1}{a}\right) = 0 \Leftrightarrow \sum_{cyc} \left(\frac{a^2 - 1}{a}\right) = 0.$$

Rewrite the inequality to the following form

$$\sum_{cyc} \left(2a - \sqrt{a^2 + 3}\right) \geq 0 \Leftrightarrow \sum_{cyc} \frac{a^2 - 1}{2a + \sqrt{a^2 + 3}} \geq 0.$$

How to continue? The idea is to apply **Chebyshev** inequality for these sequences:

$$(a^2-1, b^2-1, c^2-1, d^2-1)\ ;\ \left(\frac{1}{2a + \sqrt{a^2 + 3}}, \frac{1}{2b + \sqrt{b^2 + 3}}, \frac{1}{2c + \sqrt{c^2 + 3}}, \frac{1}{2d + \sqrt{d^2 + 3}}\right).$$

However, it's a futile idea because the first sequence is increasing but the second sequence is decreasing, and therefore the sign becomes reversed if we apply **Chebyshev**.

Hope is not lost! Noticing that $\sum_{cyc}\left(\frac{a^2-1}{a}\right)=0$, we will transform the inequality to the form

$$\sum_{cyc}\frac{a^2-1}{a}\cdot\frac{a}{2a+\sqrt{a^2+3}}\geq 0.$$

Suppose $a\geq b\geq c\geq d$. With the identity $\frac{a}{2a+\sqrt{a^2+3}}=\frac{1}{2+\sqrt{1+\frac{3}{a^2}}}$, we we see that

$$\left(\frac{a^2-1}{a},\frac{b^2-1}{b},\frac{c^2-1}{c},\frac{d^2-1}{d}\right),$$

and

$$\left(\frac{a}{2a+\sqrt{a^2+3}},\frac{b}{2b+\sqrt{b^2+3}},\frac{c}{2c+\sqrt{c^2+3}},\frac{d}{2d+\sqrt{d^2+3}}\right)$$

are two increasing sequences. So, according to **Chebyshev** inequality, we conclude

$$\sum_{cyc}\left(\frac{a^2-1}{a}\right)\cdot\left(\frac{a}{2a+\sqrt{a^2+3}}\right)\geq\frac{1}{4}\left(\sum_{cyc}\frac{a^2-1}{a}\right)\left(\sum_{cyc}\frac{a}{2a+\sqrt{a^2+3}}\right)=0.$$

This ends the proof. Equality holds for $a=b=c=d=1$.

∇

What is the key feature of this simple solution? It is the step of dividing both numerators and denominators of fractions by suitable coefficients in order to fit the condition in **Chebyshev** inequality and bond it with the hypothesis. According to this solution, we can build a general approach as follow

★ *Suppose that we need to prove the inequality (represented as a sum of fractions)*

$$\frac{x_1}{y_1}+\frac{x_2}{y_2}+...+\frac{x_n}{y_n}\geq 0$$

in which $x_1,x_2,...,x_n$ are real numbers and $y_1,y_2,...,y_n$ are positive real numbers. Generally, every inequality can be transformed into this form if a certain fraction has a negative denominator, we will multiply both its numerator and denominator by -1. Then we will find a new sequence of positive real numbers $(a_1,a_2,...,a_n)$ such that the sequence

$$(a_1x_1,a_2x_2,...,a_nx_n)$$

is increasing but the sequence

$$(a_1y_1,a_2y_2,...,a_ny_n)$$

is decreasing. After applying Chebyshev inequality

$$\sum_{i=1}^{n} \frac{x_i}{y_i} \geq \frac{1}{n} \Big(\sum_{cyc} a_i x_i \Big) \sum_{cyc} \Big(\frac{1}{a_i y_i} \Big),$$

it will remain to prove that

$$\sum_{i=1}^{n} a_i x_i \geq 0.$$

∇

Why is this approach advantageous? Because it get rid of fractions in the inequality. Even a suitable choice that makes $a_1x_1 + a_2x_2 + ... + a_nx_n = 0$ can help finish the proof immediately. In fact, many problems can be solved in this simple way. Right now, let's go further with the following examples:

Example 3.2.2. *Suppose a, b, c are positive real numbers with sum 3. Prove that*

$$\frac{1}{c^2 + a + b} + \frac{1}{a^2 + b + c} + \frac{1}{b^2 + a + c} \leq 1.$$

SOLUTION. The inequality is equivalent to

$$\sum_{cyc} \left(\frac{1}{c^2 - c + 3} - \frac{1}{3} \right) \geq 0 \Leftrightarrow \sum_{cyc} \left(\frac{a(a-1)}{a^2 - a + 3} \right) \geq 0$$

or

$$\sum_{cyc} \left(\frac{a-1}{a - 1 + \frac{3}{a}} \right) \geq 0.$$

According to **Chebyshev** inequality and the hypothesis that $a + b + c = 3$, it suffices to prove that if $a \geq b$ then $a - 1 + \frac{3}{a} \leq b - 1 + \frac{3}{b}$ or $(a - b)(ab - 3) \leq 0$. It's obviously true because $ab \leq \frac{1}{4}(a + b)^2 \leq \frac{9}{4} < 3$. Equality holds for $a = b = c = 1$.

∇

Example 3.2.3. *Let a, b, c be positive real numbers and $0 \leq k \leq 2$. Prove that*

$$\frac{a^2 - bc}{b^2 + c^2 + ka^2} + \frac{b^2 - ca}{c^2 + a^2 + kb^2} + \frac{c^2 - ab}{a^2 + b^2 + kc^2} \geq 0.$$

(Pham Kim Hung)

SOLUTION. Although this problem can be solved in the same way as example 2.1.1 is solved, we can use **Chebyshev** inequality to give a simpler solution. Notice that if $a \geq b$ then for all positive real c, we have $(a^2 - bc)(b + c) \geq (b^2 - ca)(c + a)$, and

$$(b^2 + c^2 + ka^2)(b + c) - (c^2 + a^2 + kb^2)(c + a) = (b - a) \left(\sum_{cyc} a^2 - (k - 1) \sum_{cyc} bc \right) \leq 0.$$

Having these results, we will rewrite the inequality into the following form

$$\sum_{cyc} \frac{(a^2-bc)(b+c)}{(b+c)(b^2+c^2+ka^2)} \ge 0,$$

which is obvious by **Chebyshev** inequality because $\sum_{cyc}(a^2-bc)(b+c) = 0$.

$$\nabla$$

Example 3.2.4. *Let a, b, c be positive real numbers. Prove that*

$$\sqrt{a^2+8bc} + \sqrt{b^2+8ca} + \sqrt{c^2+8ab} \le 3(a+b+c).$$

SOLUTION. Rewrite the inequality in the following form

$$\sum_{cyc}\left(3a - \sqrt{a^2+8bc}\right) \ge 0 \Leftrightarrow \sum_{cyc} \frac{a^2-bc}{3a+\sqrt{a^2+8bc}} \ge 0$$

or

$$\sum_{cyc} \frac{(a^2-bc)(b+c)}{(b+c)\left(3a+\sqrt{a^2+8bc}\right)} \ge 0.$$

According to **Chebyshev** inequality, it remains to prove that : if $a \ge b$ then

$$(b+c)\left(3a+\sqrt{a^2+8bc}\right) \le (a+c)\left(3b+\sqrt{b^2+8ca}\right)$$

$$\Leftrightarrow \; (b+c)\sqrt{a^2+8bc} - (a+c)\sqrt{b^2+8ca} \le 3c(b-a).$$

We make a transformation by conjugating:

$$(b+c)\sqrt{a^2+8bc} - (a+c)\sqrt{b^2+8ca} = \frac{(b+c)^2(a^2+8bc) - (a+c)^2(b^2+8ac)}{(b+c)\sqrt{a^2+8bc} + (a+c)\sqrt{b^2+8ca}}$$

$$= \frac{c(b-a)\left[\,8a^2+8b^2+8c^2+15c(a+b)+6ab\,\right]}{(b+c)\sqrt{a^2+8bc} + (a+c)\sqrt{b^2+8ca}}.$$

It remains to prove that

$$8a^2+8b^2+8c^2+15c(a+b)+6ab \ge 3\left((b+c)\sqrt{a^2+8bc} + (a+c)\sqrt{b^2+8ca}\right),$$

which follows immediately from **AM-GM** since

$$\text{RHS} \le \frac{3}{2}\left((b+c)^2 + (a^2+8bc) + (a+c)^2 + (b^2+8ca)\right)$$

$$= 3(a^2+b^2+c^2+5bc+5ac) \le \text{LHS}.$$

Comment. Here is a similar example

★ *Let a, b, c be positive real numbers. Prove that*

$$\frac{a^2-bc}{\sqrt{7a^2+2b^2+2c^2}} + \frac{b^2-ca}{\sqrt{7b^2+2c^2+2a^2}} + \frac{c^2-ab}{\sqrt{7c^2+2a^2+2b^2}} \ge 0.$$

$$\nabla$$

Example 3.2.5. *Let a, b, c, d be positive real numbers such that $a^2+b^2+c^2+d^2=4$. Prove that*

$$\frac{1}{5-a}+\frac{1}{5-b}+\frac{1}{5-c}+\frac{1}{5-d}\le 1.$$

(Pham Kim Hung)

SOLUTION. The inequality is equivalent to

$$\sum_{cyc}\left(\frac{1}{5-a}-\frac{1}{4}\right)\le 0 \Leftrightarrow \sum_{cyc}\frac{a-1}{5-a}\le 0$$

$$\Leftrightarrow \sum_{cyc}\frac{(a-1)(a+1)}{(5-a)(a+1)}\le 0 \Leftrightarrow \sum_{cyc}\frac{a^2-1}{4a-a^2+5}\le 0.$$

Notice that $\sum_{cyc}(a^2-1)=0$, so by **Chebyshev** inequality, it is enough to prove that if $a\ge b$ then

$$4a-a^2+5\ge 4b-b^2+5.$$

This condition is reduced to $a+b\le 4$, which is obvious because $a^2+b^2\le 4$. Equality holds for $a=b=c=d=1$.

∇

Example 3.2.6. *Let $a_1, a_2, ..., a_n$ be positive real numbers satisfying*

$$a_1+a_2+...+a_n=\frac{1}{a_1}+\frac{1}{a_2}+...+\frac{1}{a_n}.$$

Prove that the following inequality holds

$$\frac{1}{n^2+a_1^2-1}+\frac{1}{n^2+a_2^2-1}+...+\frac{1}{n^2+a_n^2-1}\ge\frac{1}{a_1+a_2+...+a_n}.$$

(Pham Kim Hung)

SOLUTION. WLOG, we may assume that $a_1\ge a_2\ge ...\ge a_n$. The hypothesis is equivalent to:

$$\frac{1-a_1^2}{a_1}+\frac{1-a_2^2}{a_2}+...+\frac{1-a_n^2}{a_n}=0\ (*)$$

Denote $S=\sum_{i=1}^{n}a_i$ and $k=n^2-1$. According to (*), the inequality can be rewritten as

$$\frac{1-a_1}{k+a_1^2}+\frac{1-a_2}{k+a_2^2}+...+\frac{1-a_n}{k+a_n^2}\ge\frac{n-S}{S}$$

$$\Leftrightarrow \sum_{i=1}^{n}\frac{1-a_i^2}{a_i}\left[\frac{a_i}{(1+a_i)(k+a_i^2)}-\frac{a_i}{(1+a_i)S}\right]\ge 0.$$

For each $i \neq j$ and $i, j \in \{1, 2, ..., n\}$, we denote

$$\begin{aligned} S_{ij} &= \left[\frac{a_i}{(1+a_i)(k+a_i^2)} - \frac{a_i}{(1+a_i)S}\right] - \left[\frac{a_j}{(1+a_j)(k+a_j^2)} - \frac{a_j}{(1+a_j)S}\right] \\ &= \frac{a_j - a_i}{(1+a_i)(1+a_j)}\left[\frac{a_i a_j(a_i+a_j+1)-k}{(a_i^2+k)(a_j^2+k)} + \frac{1}{S}\right]. \end{aligned}$$

If $S_{ij} \leq 0$ for all $1 \leq i < j \leq n$ $(i, j \in \mathbb{N})$ then by **Chebyshev** inequality, we conclude

$$\sum_{i=1}^{n} \frac{1-a_i^2}{a_i}\left[\frac{a_i}{(1+a_i)(k+a_i^2)} - \frac{a_i}{(1+a_i)S}\right]$$

$$\geq \frac{1}{n}\left[\sum_{i=1}^{n}\frac{1-a_i^2}{a_i}\right]\left[\sum_{i=1}^{n}\left(\frac{a_i}{(1+a_i)(k+a_i^2)} - \frac{a_i}{(1+a_i)S}\right)\right] = 0.$$

Otherwise, suppose that there exist two indexes $i < j$ such that $S_{ij} \geq 0$ or

$$\frac{a_i a_j(a_i+a_j+1)-k}{(a_i^2+k)(a_j^2+k)} + \frac{1}{S} \leq 0.$$

This condition implies that

$$\frac{1}{S} \leq \frac{k - a_i a_j(a_i+a_j+1)}{(a_i^2+k)(a_j^2+k)} \leq \frac{1}{k+a_i^2} + \frac{1}{k+a_j^2} < \sum_{i=1}^{n}\frac{1}{k+a_i^2}.$$

This ends the proof. The equality holds for $a_1 = a_2 = ... = a_n = 1$.

∇

Why does the **Chebyshev** associate technique stand out from other ways of applying **Chebyshev**? Perhaps its wide application is the reason. Do you think this new and surprising? Surely not. In fact, this approach is natural. I believe that you have already used it but haven't given it a common name yet. From now, on instead of thinking *intuitively* and *accidentaly*, you will use **Chebyshev** associate technique *intentionally*. Sometimes the work of multiplying numerators and denominators by suitable coefficients is quite conspicious, such as in the problems above, but sometimes it's not. Figuring out the good coefficients requires a lot of effort and therefore the "intention" to find them becomes important. Let's examine this matter through the following instances

Example 3.2.7. *Suppose that a, b, c are positive real numbers with sum 3. Prove that*

$$\frac{1}{9-ab} + \frac{1}{9-bc} + \frac{1}{9-ca} \leq \frac{3}{8}.$$

SOLUTION. Let $x = bc, y = ca, z = ab$. The inequality becomes

$$\sum_{cyc} \frac{1}{9-x} \le \frac{3}{8} \Leftrightarrow \sum_{cyc} \frac{1-x}{9-x} \ge 0.$$

Suppose that a_x, a_y, a_z are the coefficients we are looking for. We will rewrite the inequality to

$$\sum_{cyc} a_x(1-x) \cdot \frac{1}{a_x(9-x)} \ge 0.$$

The numbers (a_x, a_y, a_z) must fulfill two conditions: first, among two sequences $(a_x(1-x), a_y(1-y), a_z(1-z))$ and $(a_x(9-x), a_y(9-y), a_z(9-z))$, one is increasing and another is decreasing (1); second, $\sum\limits_{cyc} a_x(1-x) \ge 0$ (2).

Let's do some tests. We first choose $a_x = 1+x, a_y = 1+y, a_z = 1+z$. In this case, condition (1) is satisfied but condition (2) is false because

$$\sum a_x(1-x) = 3 - \sum_{cyc} x^2 \le 0.$$

We then choose $a_x = 8+x, a_y = 8+y, a_z = 8+z$. This time, condition (2) is satisfied (you can check it easily) but condition (1) is not always true. Fortunately, everything is fine if we choose $a_x = 6+x, a_y = 6+y, a_z = 6+z$. In this case, it's obvious that if $x \ge y \ge z$ then $a_x(1-x) \ge a_y(1-y) \ge a_z(1-z)$ and $a_x(9-x) \le a_y(9-y) \le a_z(9-z)$. It remains to prove that

$$\sum_{cyc} a_x(1-x) \ge 0 \Leftrightarrow 5\left(\sum_{cyc} ab\right) + \left(\sum_{cyc} ab\right)^2 \le 18 + 6abc.$$

By **AM-GM**, we have

$$\prod_{cyc}(3-2a) = \prod_{cyc}(a+b-c) \le abc.$$

This can be reduced to $9 + 3abc \ge 4\sum\limits_{cyc} ab$. Replacing $3abc \ge 4\sum\limits_{cyc} ab - 9$ in the above inequality, it becomes

$$5\left(\sum_{cyc} ab\right) + \left(\sum_{cyc} ab\right)^2 \le 8\left(\sum_{cyc} ab\right) \Leftrightarrow \sum_{cyc} ab \le 3,$$

which is obvious because $a+b+c=3$. Equality holds for $a=b=c=1$.

∇

Example 3.2.8. *Let a, b, c be positive real numbers such that $a^4+b^4+c^4=3$. Prove that*

$$\frac{1}{4-ab}+\frac{1}{4-bc}+\frac{1}{4-ca}\le 1.$$

(Moldova TST 2005)

SOLUTION. Let $x=ab, y=ac$ and $z=bc$. The inequality is equivalent to

$$\frac{1-x}{4-x}+\frac{1-y}{4-y}+\frac{1-z}{4-z}\ge 0$$

$$\Leftrightarrow \quad \frac{1-x^2}{4+3x-x^2}+\frac{1-y^2}{4+3y-y^2}+\frac{1-z^2}{4+3z-z^2}\ge 0$$

Notice that $a^4+b^4+c^4=3$ so $x^2+y^2+z^2\le 3$ and therefore if $x\ge y\ge z$ then

$$1-x^2\le 1-y^2\le 1-z^2 \quad ; \quad 4+3x-x^2\ge 4+3y-y^2\ge 4+3z-z^2 \quad ;$$

thus by **Chebyshev** inequality, we obtain

$$\sum_{cyc}\frac{1-x^2}{4+3x-x^2}\ge\frac{1}{3}\left(\sum_{cyc}(1-x^2)\right)\left(\sum_{cyc}\frac{1}{4+3x-x^2}\right)\ge 0$$

because $\sum_{cyc} x\le 3$ and $\sum_{cyc} x^2\le 3$. Equality holds for $a=b=c=1$.

∇

Example 3.2.9. *Let $a_1, a_2, ..., a_n$ be positive real numbers such that*

$$a_1+a_2+...+a_n=\frac{1}{a_1}+\frac{1}{a_2}+...+\frac{1}{a_n}.$$

Prove the following inequality

$$\frac{1}{n-1+a_1^2}+\frac{1}{n-1+a_2^2}+...+\frac{1}{n-1+a_n^2}\le 1.$$

(Pham Kim Hung)

SOLUTION. Rewrite the inequality to the following from

$$\sum_{i=1}^{n}\left(\frac{1}{n-1+a_i^2}-\frac{1}{n}\right)\le 0$$

or equivalently

$$\sum_{i=1}^{n}\frac{a_i^2-1}{n-1+a_i^2}\ge 0.$$

Assume that $a_1 \geq a_2 \geq \ldots \geq a_n$. According to the hypothesis, we have

$$\sum_{i=1}^{n} \frac{a_i^2 - 1}{a_i} = 0.$$

Moreover, notice that

$$\frac{a_i}{n-1+a_i^2} - \frac{a_j}{n-1+a_j^2} = \frac{(n-1-a_ia_j)(a_i-a_j)}{(n-1+a_i^2)(n-1+a_j^2)}.$$

So, in case $a_ia_j \leq n-1 \ \forall i \neq j$, we can conclude that

$$\sum_{i=1}^{n} \frac{a_i^2-1}{n-1+a_i^2} \geq \frac{1}{n}\left(\sum_{i=1}^{n} \frac{a_i^2-1}{a_i}\right)\left(\sum_{i=1}^{n} \frac{a_i}{n-1+a_i^2}\right) = 0.$$

It suffices to consider the remaining case $a_1a_2 \geq n-1$. For $n \geq 3$, **Cauchy-Schwarz** inequality shows that

$$\frac{a_1^2}{n-1+a_1^2} + \frac{a_2^2}{n-1+a_2^2} \geq \frac{(a_1+a_2)^2}{2(n-1)+a_1^2+a_2^2} \geq 1$$

$$\Rightarrow \sum_{i=1}^{n} \frac{a_i^2}{n-1+a_i^2} \geq 1 \Rightarrow \sum_{i=1}^{n} \frac{1}{n-1+a_i^2} \leq 1.$$

For $n = 1$ and $n = 2$, the inequality becomes an equality. For $n \geq 3$, the equality holds if and only if $a_1 = a_2 = \ldots = a_n = 1$.

∇

Chapter 4

Inequalities with Convex Functions

The convex function is an important concept and plays an important role in many fields of Mathematics. Although convex functions always pertain to advanced theories, this book will try to give you the most fundamental knowledge of this kind of functions so that it can be easily understood by a high-school student and it can be used in inequalities. This section includes two smaller parts: **Jensen** inequality and inequalities with bounded variables.

4.1 Convex functions and Jensen inequality

Definition 1. *Suppose that f is a one-variable function defined on $[a, b] \subset \mathbb{R}$. f is called a convex function on $[a, b]$ if and only if for all $x, y \in [a, b]$ and for all $0 \le t \le 1$, we have*

$$tf(x) + (1-t)f(y) \ge f\left(tx + (1-t)y\right).$$

Theorem 5. *If $f(x)$ is a real function defined on $[a, b] \subset \mathbb{R}$ and $f''(x) \ge 0$ $\forall x \in [a, b]$ then $f(x)$ is a convex function on $[a, b]$.*

PROOF. We will prove that for all $x, y \in [a, b]$ and for all $0 \le t \le 1$

$$tf(x) + (1-t)f(y) \ge f\left(tx + (1-t)y\right).$$

Indeed, suppose that t and y are constant. Denote

$$g(x) = tf(x) + (1-t)f(y) - f\left(tx + (1-t)y\right).$$

By differentiating,

$$g'(x) = tf'(x) - tf'(tx + (1-t)y).$$

Notice that $f''(x) \geq 0$ for all $x \in [a,b]$, thus $f'(x)$ is an increasing function on $[a,b]$. This gives that $g'(x) \geq 0$ if $x \geq y$ and $g'(x) \leq 0$ if $x \leq y$. That means $g(x) \geq g(y) = 0$.

∇

Theorem 6 (Jensen inequality). *Suppose that f is a convex function on $[a,b] \subset \mathbb{R}$. For all $x_1, x_2, ..., x_n \in [a,b]$, we have*

$$f(x_1) + f(x_2) + ... + f(x_n) \geq nf\left(\frac{x_1 + x_2 + ... + x_n}{n}\right).$$

∇

If you've never read any material regarding convex functions, or if you've never seen the definition of a convex function, the following lemma seems to be very useful and practical (although it can be obtained directly from **Jensen** inequality)

Lemma 1. *Suppose that a real function $f : [a,b] \to \mathbb{R}$ satisfies the condition*

$$f(x) + f(y) \geq 2f\left(\frac{x+y}{2}\right) \quad \forall x, y \in [a,b],$$

then for all $x_1, x_2, ..., x_n \in [a,b]$, the following inequality holds

$$f(x_1) + f(x_2) + ... + f(x_n) \geq nf\left(\frac{x_1 + x_2 + ... + x_n}{n}\right).$$

PROOF. We use **Cauchy** induction to solve this lemma. By hypothesis, the inequality holds for $n = 2$, therefore it holds for every number n that is a power of 2. It's enough to prove that if the inequality holds for $n = k + 1 (k \in \mathbb{N}, k \geq 2)$ then it will hold for $n = k$. Indeed, suppose that it's true for $n = k + 1$. Denote $x = x_1 + x_2 + ... + x_k$ and take $x_{k+1} = \frac{x}{k}$. By the inductive hypothesis, we have

$$f(x_1) + f(x_2) + ... + f(x_k) + f\left(\frac{x}{k}\right) \geq (k+1)f\left(\frac{x + \frac{x}{k}}{k+1}\right) = (k+1)f\left(\frac{x}{k}\right).$$

That finishes the proof.

∇

The result above can can directly infered from **Jensen** inequality because according to the definition, every convex function f satisfies $(t = 1/2)$

$$f(x) + f(y) \geq 2f\left(\frac{x+y}{2}\right).$$

Obviously, if we change the condition $f(x)+f(y)\geq 2f\left(\dfrac{x+y}{2}\right)$ $\forall x,y\in[a,b]$ to $f(x)+f(y)\leq 2f\left(\dfrac{x+y}{2}\right)$ $\forall x,y\in[a,b]$, then the sign of the inequality is reversed

$$f(x_1)+f(x_2)+...+f(x_n)\leq nf\left(\frac{x_1+x_2+...+x_n}{n}\right).$$

Lemma 2. *Suppose that the real function $f:[a,b]\to\mathbb{R}^+$ satisfies the condition*

$$f(x)+f(y)\geq 2f(\sqrt{xy})\ \forall x,y\in[a,b],$$

then for all $x_1,x_2,...,x_n\in[a,b]$, the following inequality holds

$$f(x_1)+f(x_2)+...+f(x_n)\geq nf(\sqrt[n]{x_1x_2...x_n}).$$

∇

The proof of this lemma is completely similar to that of lemma 1 and therefore it won't be shown here. Notice that this lemma is quite widely applied and it is related to the **AM-GM** inequality of course.

Theorem 7 (Weighted Jensen inequality). *Suppose that $f(x)$ is a real function defined on $[a,b]\subset\mathbb{R}$ and $x_1,x_2,...,x_n$ are real numbers on $[a,b]$. For all non-negative real numbers $a_1,a_2,...,a_n$ which sum up to 1, the following inequality holds*

$$a_1f(x_1)+a_2f(x_2)+...+a_nf(x_n)\geq f(a_1x_1+a_2x_2+...+a_nx_n).$$

Jensen *inequality is a particular case of this theorem for $a_1=a_2=...=a_n=1/n$.*

Let's consider a more elementary version of this theorem as follow

Lemma 3. *Let $a_1,a_2,...,a_n$ be non-negative real numbers with sum 1 and $x_1,x_2,...,x_n$ be real numbers in $[a,b]$. Let $f(x)$ be a real function defined on $[a,b]$. The inequality*

$$a_1f(x_1)+a_2f(x_2)+...+a_nf(x_n)\geq f(a_1x_1+a_2x_2+...+a_nx_n)$$

is true for every positive integer n and for every real numbers x_i, a_i, $i=1,2,\cdots,n$ if and only if it's true in the case $n=2$.

∇

To prove lemma 3 as well as the weighted **Jensen** inequality, we use the same method as in the proof of lemma 1. The great advantage of lemma 1, 2 and 3 is that is allows one to use the convex-function method even if one knows nothing about the convexity of a function. Following is an obvious corollary

Corollary 3.

a, The conclusion in lemma 1 is still true if we change the expression of the arithmetic mean by any other average form of $x_1, x_2, ..., x_n$*; for example, geometric mean or harmonic mean, etc.*

b, If the sign of the inequality for two numbers is reversed, the sign of the inequality for n numbers is reversed, too.

Jensen is a classical inequality. In the next chapters, we will continue discussing this inequality in relationship with **Karamata** inequality, a stronger result. Now let's continue with some applications of **Jensen** inequality.

Example 4.1.1. *Suppose that* $x_1, x_2, ..., x_n$ *are positive real numbers and* $x_1, x_2, ..., x_n \geq 1$. *Prove that*

$$\frac{1}{1+x_1} + \frac{1}{1+x_2} + ... + \frac{1}{1+x_n} \leq \frac{n}{1+\sqrt[n]{x_1 x_2 ... x_n}}.$$

(IMO Shortlist)

SOLUTION. According to lema 2, it's enough to prove that

$$\frac{1}{1+a^2} + \frac{1}{1+b^2} \leq \frac{2}{1+ab}\ \forall a, b \geq 1.$$

We can reduce this inequality to $(a-b)^2(1-ab) \leq 0$, which is obvious.

∇

Example 4.1.2. *Let* $a_1, a_2, ..., a_n$ *be real numbers lying in* $(1/2, 1]$. *Prove that*

$$\frac{a_1 a_2 ... a_n}{(a_1 + a_2 + ... + a_n)^n} \geq \frac{(1-a_1)(1-a_2)...(1-a_n)}{(n - a_1 - a_n - .. - a_n)^n}.$$

SOLUTION. The inequality is equivalent to

$$\sum_{i=1}^{n} \left(\ln a_i - \ln(1-a_i) \right) \geq n \ln \left(\sum_{i=1}^{n} a_i \right) - n \ln \left(n - \sum_{i=1}^{n} a_i \right).$$

Notice that the function $f(x) = \ln x - \ln(1-x)$, has the second derivative

$$f''(x) = \frac{-1}{x^2} + \frac{1}{(1-x)^2} \geq 0\ (x \in (1/2, 1]).$$

Therefore f is a convex function. By **Jensen** inequality, we have the desired result.

∇

In comparison with other basic inequalities such as **AM-GM Cauchy-Schwarz** or **Chebyshev** apparently, **Jensen** inequality is restricted to a separate world. **Jensen** inequality is so rarely used because people always think that it is not strong enough for difficult problems. However, there is an undiscoved field of inequalities where **Jensen** inequality becomes very effective and always gives us unexpected solutions.

Example 4.1.3. *Let a, b, c be positive real numbers. Prove that*

$$\frac{a}{\sqrt{a^2+8bc}}+\frac{b}{\sqrt{b^2+8ac}}+\frac{c}{\sqrt{c^2+8ab}} \geq 1.$$

(IMO 2001, A2)

SOLUTION. Although this problem has been solved using **Hölder**, a proof by **Jensen's** inequality is very nice, too. WLOG, we may assume that $a+b+c=1$ normalize. Because $f(x)=\frac{1}{\sqrt{x}}$ is a convex function, we obtain from **Jensen's** inequality that:

$$a \cdot f(a^2+8bc)+b \cdot f(b^2+8ca)+c \cdot f(c^2+8ab) \geq f(M)$$

in which $M=\sum_{cyc} a(a^2+8bc)=24abc+\sum_{cyc} a^3$. It remains to prove that $f(M) \geq 1$ or $M \leq 1$ or

$$24abc+\sum_{cyc} a^3 \leq \left(\sum_{cyc} a\right)^3 \Leftrightarrow \sum c(a-b)^2 \geq 0.$$

This last inequality is obvious. Equality holds for $a=b=c$.

∇

Example 4.1.4. *Let a, b, c, d be positive numbers with sum 4. Prove that*

$$\frac{a}{b^2+b}+\frac{b}{c^2+c}+\frac{c}{d^2+d}+\frac{d}{a^2+a} \geq \frac{8}{(a+c)(b+d)}.$$

SOLUTION. Denote $f(x)=\frac{1}{x(x+1)}$, then f is a convex function if $x>0$. According to **Jensen** inequality, we have

$$\frac{a}{4} \cdot f(b)+\frac{b}{4} \cdot f(c)+\frac{c}{4} \cdot f(d)+\frac{d}{4} \cdot f(a) \geq f\left(\frac{ab+bc+cd+da}{4}\right),$$

which can be rewritten as

$$\sum_{cyc} \frac{a}{b^2+b} \geq \frac{64}{(ab+bc+cd+da)^2+4(ab+bc+cd+da)}$$

It remains to prove that

$$\frac{64}{(ab+bc+cd+da)^2+4(ab+bc+cd+da)} \geq \frac{8}{ab+bc+cd+da}$$

$$\Leftrightarrow\ ab+bc+cd+da \leq 4 \ \Leftrightarrow\ (a-b+c-d)^2 \geq 0.$$

Equality holds for $a=b=c=d=1$.

$$\nabla$$

Example 4.1.5. *Suppose that a,b,c are positive real numbers. Prove that*

$$\sqrt{\frac{a}{a+b}}+\sqrt{\frac{b}{b+c}}+\sqrt{\frac{c}{c+a}} \leq \frac{3}{\sqrt{2}}.$$

(Vasile Cirtoaje)

SOLUTION. Notice that $f(x)=\sqrt{x}$ is a concave function. According to **Jensen** inequality, we have

$$\sum_{cyc}\sqrt{\frac{a}{a+b}}=\sum_{cyc}\frac{a+c}{2(a+b+c)}\cdot\sqrt{\frac{4a(a+b+c)^2}{(a+b)(a+c)^2}}$$

$$\leq\sqrt{\sum_{cyc}\frac{a+c}{2(a+b+c)}\cdot\frac{4a(a+b+c)^2}{(a+b)(a+c)^2}}=\sqrt{\sum_{cyc}\frac{2a(a+b+c)}{(a+c)(b+c)}}.$$

It remains to prove that

$$\sum_{cyc}\frac{a(a+b+c)}{(a+c)(b+c)} \leq \frac{9}{4}.$$

After expanding, the inequality becomes

$$8\left(\sum_{cyc}ab\right)\left(\sum_{cyc}a\right) \geq 9\prod(a+b) \ \Leftrightarrow\ \sum_{cyc}c(a-b)^2 \geq 0.$$

$$\nabla$$

Example 4.1.6. *Let a,b,c be non-negative real numbers. Prove that*

$$\frac{a}{\sqrt{4b^2+bc+4c^2}}+\frac{b}{\sqrt{4c^2+ca+4a^2}}+\frac{c}{\sqrt{4a^2+ab+4b^2}} \geq 1.$$

(Pham Kim Hung, Vo Quoc Ba Can)

SOLUTION. We may assume that $a+b+c=1$. Because $f(x)=\frac{1}{\sqrt{x}}$ is a convex function, according to **Jensen** inequality, we have

$$a\cdot f(4b^2+bc+4c^2)+b\cdot f(4c^2+ca+4a^2)+c\cdot f(4a^2+ab+4b^2)\geq f(M),$$

where

$$M=a(4b^2+bc+4c^2)+b(4c^2+ca+4a^2)+c(4a^2+ab+4b^2)=4\sum_{cyc}ab(a+b)+3abc.$$

It suffices to prove that $f(M)\geq 1$ or $M\leq 1$. It's certainly true because

$$1-M=\left(\sum_{cyc}a\right)^3-4\sum_{cyc}ab(a+b)-3abc=\sum_{cyc}a^3-\sum_{cyc}ab(a+b)+3abc$$
$$=\prod_{cyc}abc-\prod_{cyc}(a+b-c)\geq 0.$$

Equality holds for $a=b=c$ and $a=0,b=c$ up to permutation.

∇

Example 4.1.7. *Let a,b,c be positive real numbers. Prove that*

$$\sqrt{\frac{a}{4a+4b+c}}+\sqrt{\frac{b}{4b+4c+a}}+\sqrt{\frac{c}{4c+4a+b}}\leq 1.$$

(Pham Kim Hung)

SOLUTION. Notice that $f(x)=\sqrt{x}$ is a concave function, therefore by **Jensen** inequality we have

$$\sum_{cyc}\sqrt{\frac{a}{4a+4b+c}}=\sum_{cyc}\frac{(4a+4c+b)}{9(a+b+c)}\cdot\sqrt{\frac{81a(a+b+c)^2}{(4a+4b+c)(4b+4c+a)^2}}$$
$$\leq\sqrt{\sum_{cyc}\frac{(4a+4c+b)}{9(a+b+c)}\cdot\frac{81a(a+b+c)^2}{(4a+4b+c)(4a+4c+b)^2}}$$
$$=\sqrt{\sum_{cyc}\frac{9a(a+b+c)}{(4a+4b+c)(4a+4c+b)}}.$$

WLOG, we may assume that $a+b+c=1$. It remains to prove

$$\sum_{cyc}\frac{9a(a+b+c)}{(4a+4b+c)(4a+4c+b)}\leq 1$$

or equivalently

$$9\left(\sum_{cyc} a^2 + 8\sum_{cyc} ab\right) \le \prod_{cyc}(4-3a) \Leftrightarrow 18\sum_{cyc} ab + 27abc \le 7,$$

which is certainly true because $\sum_{cyc} ab \le \frac{1}{3}$ and $abc \le \frac{1}{27}$.

$$\nabla$$

Example 4.1.8. *Let a, b, c be positive real numbers such that $a^2+b^2+c^2=3$. Prove that*

$$\sqrt{\frac{a}{a^2+b^2+1}} + \sqrt{\frac{b}{b^2+c^2+1}} + \sqrt{\frac{c}{c^2+a^2+1}} \le \sqrt{3}.$$

(Pham Kim Hung)

SOLUTION. Applying **Jensen** inequality for the concave function $f(x) = \sqrt{x}$, we have

$$\sum_{cyc}\sqrt{\frac{a}{a^2+b^2+1}} = \sum_{cyc}\frac{a^2+c^2+1}{3(a^2+b^2+c^2)} \cdot \sqrt{\frac{9a(a^2+b^2+c^2)^2}{(a^2+b^2+1)(a^2+c^2+1)^2}}$$

$$\le \sqrt{\sum_{cyc}\frac{a^2+c^2+1}{3(a^2+b^2+c^2)} \cdot \frac{9a(a^2+b^2+c^2)^2}{(a^2+b^2+1)(a^2+c^2+1)^2}}$$

$$= \sqrt{\sum_{cyc}\frac{3a(a^2+b^2+c^2)}{(a^2+b^2+1)(a^2+c^2+1)}}.$$

It remains to prove that

$$\sum_{cyc}\frac{a}{(a^2+b^2+1)(a^2+c^2+1)} \le \frac{1}{3} \Leftrightarrow 3\sum_{cyc} a(b^2+c^2+1) \le \prod_{cyc}(4-a^2).$$

This inequality can be reduced to

$$12\sum_{cyc} a - 3\sum_{cyc} a^3 \le 34 - a^2b^2c^2 - 2\sum_{cyc} a^4.$$

By **AM-GM**, we have $a^2b^2c^2 \le 1$, so it's sufficient to prove that

$$\sum_{cyc}(2a^4 - 3a^3 + 12a - 11) \le 0 \Leftrightarrow \sum_{cyc}(a^2-1)\left(2a^2 - 3a + 2 + \frac{9}{a+1}\right) \le 0.$$

Because this last inequality is symmetric, we can assume that $a \ge b \ge c$. Denote

$$S_a = 2a^2 - 3a + 2 + \frac{9}{a+1}, S_b = 2b^2 - 3b + 2 + \frac{9}{b+1}, S_c = 2c^2 - 3c + 2 + \frac{9}{c+1}.$$

If $b \leq 1$ then $a+b \leq 1+\sqrt{2}$ and $(a+1)(b+1) \leq 2\left(1+\sqrt{2}\right)$. It implies

$$S_a - S_b = 2(a+b) - 3 - \frac{9}{(a+1)(b+1)} \leq 0.$$

Certainly $S_b - S_c \leq 0$, so we conclude that $S_a \leq S_b \leq S_c$. By **Chebyshev** inequality, we obtain

$$\sum_{cyc}(a^2-1)S_a \leq \frac{1}{3}\left(\sum_{cyc}(a^2-1)\right)\left(\sum_{cyc} S_a\right) = 0.$$

If $b \geq 1$ then we also have $S_a - S_c \leq 0$ and $S_b - S_c \leq 0$ (because $c \leq 1$). It implies

$$\sum_{cyc}(a^2-1)S_a = (a^2-1)(S_a - S_c) + (b^2-1)(S_b - S_c) \leq 0.$$

Equality holds for $a = b = c = 1$.

∇

In fact, the weighted **Jensen** inequality has, to some extent, much secrets. It's still rarely used nowadays but once used, it always shows a wonderful solution. Problems and solutions above, hopefully, convey to you a certain way of using this special approach; it should be contemplated more by yourself.

In the following pages, we will discuss a new way of applying convex functions to inequalities. We will use convex functions to handle inequalities whose variables are restricted in a fixed range $[a, b]$.

4.2 Convex Functions and Inequalities with Variables Restricted to an Interval

In some inequalities, variables are restricted to a certain interval. For this kind of problems, **Jensen** inequality appears to be especially strong and useful because it helps determine if the variables are equal to the boundaries or not, for the minimum of an expression to be obtained.

Example 4.2.1. *Suppose that a, b, c are positive real numbers belonging to $[1, 2]$. Prove that*

$$a^3 + b^3 + c^3 \leq 5abc.$$

(MYM 2001)

SOLUTION. Let's first give an elementary solution to this simple problem. Since $a, b, c \in [1,2]$, if $a \geq b \geq c$ then

$$\begin{aligned} a^3 + 2 \leq 5a &\Leftrightarrow (a-2)(a^2+2a-1) \leq 0 \quad (1) \\ 5a + b^3 \leq 5ab + 1 &\Leftrightarrow (b-1)(b^2+b+1-5a) \leq 0 \quad (2) \\ 5ab + c^3 \leq 5abc + 1 &\Leftrightarrow (c-1)(c^2+c+1-5ab) \leq 0 \quad (3) \end{aligned}$$

The above estimations are correct because

$$\begin{aligned} &b^2+b+1 \leq a^2+a+1 \leq 2a+a+1 \leq 5a, \\ &c^2+c+1 \leq a^2+a+1 \leq 5a \leq 5ab. \end{aligned}$$

Summing up the results (1), (2) and (3), we get the result. Equality holds if $a=2, b=c=1$ and permutations.

∇

Example 4.2.2. *Suppose that a, b, c are positive real numbers belonging to $[1,2]$. Prove that*

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \leq 10.$$

(Olympiad 30-4, Vietnam)

SOLUTION. The inequality can be rewritten as

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{b}{a}+\frac{c}{b}+\frac{a}{c} \leq 7.$$

WLOG, we may assume that $a \geq b \geq c$, then

$$(a-b)(a-c) \geq 0 \Rightarrow \begin{cases} \dfrac{a}{c}+1 \geq \dfrac{a}{b}+\dfrac{b}{c} \\ \dfrac{c}{a}+1 \geq \dfrac{c}{b}+\dfrac{b}{a}, \end{cases}$$

which implies that

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{b}+\frac{b}{a} \leq \frac{a}{c}+\frac{c}{a}+2.$$

We conclude

$$\Rightarrow \frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{b}{a}+\frac{c}{b}+\frac{a}{c} \leq 2+2\left(\frac{a}{c}+\frac{c}{a}\right) = 7-\frac{(a-2c)(2a-c)}{ac} \leq 7$$

because $2c \geq a \geq c$. Equality holds for $(a,b,c) = (2,2,1)$ or $(2,1,1)$ or permutations.

Comment. Here is the general problem. Its solution is completely similar to that of the previous problem.

★ *Suppose that $p < q$ are positive constants and $a_1, a_2, ..., a_n \in [p, q]$. Prove that*

$$(a_1 + a_2 + ... + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + ... + \frac{1}{a_n}\right) \leq n^2 + \frac{k_n(p-q)^2}{4pq},$$

where $k_n = n^2$ if n is even and $n^2 - 1$ if n is odd.

∇

The key feature of the previous solution is the intermediate estimation (estimate that if $a \geq b \geq c$ then $(b-a)(b-c) \leq 0$). Because they are so elementary and simple, a high-school student can comprehend them easily. But what happens to the following inequality? Can the previous method be used? Let's see.

Example 4.2.3. *Let $x_1, x_2, ..., x_{2005}$ be real numbers belonging to $[-1, 1]$. Find the minimum value for the following expression*

$$P = x_1x_2 + x_2x_3 + ... + x_{2004}x_{2005} + x_{2005}x_1.$$

SOLUTION. Because this inequality is cyclic, not symmetric, we can not order variables as. If we rely on the relation $(x_i - 1)(x_i + 1) \leq 0$, we won't succeed either.

By intuition, we feel that the expression will attain its maximum if in the sequence $(x_1, x_2, ..., x_{2005})$, 1 and -1 alternate. In this case

$$P = 1 \cdot (-1) + (-1) \cdot 1 + ... + (-1) \cdot 1 + 1 \cdot 1 = -2003.$$

An accurate proof of this conjecture is not so obvious. Although the following solution is simple, it's really hard if you don't have some knowledge of convex functions is never generated in your mind.

First, we notice that if $x \in [p, q]$ then each linear function $f(x) = ax + b$ or quadratic function $f(x) = x^2 + ax + b$ has the following important property

$$\max_{x \in [p,q]} f(x) = \max\{f(p), f(q)\}.$$

A linear function satisfies another condition

$$\min_{x \in [p,q]} f(x) = \min\{f(p), f(q)\}.$$

Notice that $P = P(x_1)$ is a linear function of x_1, therefore, according to properties of linear functions, P can attain the minimum if only if $x_1 \in \{-1, 1\}$. Similarly, for the other variables, we deduce that P attains the minimum if and only if $x_k \in \{-1, 1\}$ for

each $k = 1, 2, ..., 2005$. In this case, we will prove that $P \geq -2003$. Indeed, there must be at least one index k ($k \in \mathbb{N}, 1 \leq k \leq 2005$) for which $x_k x_{k+1} \geq 0$. That implies $x_k x_{k+1} = 1$ and therefore $\sum\limits_{k=1}^{n} x_i x_{i+1} \geq -2003$.

Comment. By a similar approach, we can solve a lot of inequalities of this kind such as

★ *Let $x_1, x_2, ..., x_n$ be real numbers belonging to $[-1, 1]$. Find the minimum value of*

$$P = x_1x_2x_3 + x_2x_3x_4 + ... + x_{n-1}x_nx_1 + x_nx_1x_2.$$

★ *Let $x_1, x_2, ..., x_n$ be real numbers belonging to $[0, 1]$. Prove that*

$$P = x_1(1 - x_2) + x_2(1 - x_3) + ... + x_n(1 - x_1) \leq \left[\frac{n}{2}\right].$$

∇

For what kind of functions does this approach hold? Of course, linear function is an example, but there are not all. The following lemma helps, determine a large class of such functions.

Lemma 4. *Suppose that $F(x_1, x_2, ..., x_n)$ is a real function defined on $[a, b] \times [a, b] \times \cdots \times [a, b] \subset \mathbb{R}^n$ $(a < b)$ such that for all $k \in \{1, 2, ..., n\}$, if we fix $n - 1$ variables $x_j (j \neq k)$ then $F(x_1, x_2, ..., x_n) = f(x_k)$ is a convex function of x_k. F attains its minimum at the point $(\alpha_1, \alpha_2, ..., \alpha_n)$ if and only if $\alpha_i \in \{a, b\}$ $\forall i \in \{1, 2, ..., n\}$.*

SOLUTION. In fact, we only need to prove that if $f(x)$ is a real convex function defined on $[a, b]$ then for all $x \in [a, b]$, we have

$$f(x) \leq \max\{f(a), f(b)\}.$$

Indeed, since $\{ta + (1 - t)b | t \in [0, 1]\} = [a, b]$, for all $x \in [a, b]$, there exists a number $t \in [0, 1]$ such that $x = ta + (1 - t)b$. According to the definition of a convex function, we deduce that

$$f(x) \leq tf(a) + (1 - t)f(b) \leq \max\{f(a), f(b)\}.$$

∇

With this lemma, in problems like example 4.2.3., we only need to check the convexity/concavity of a multi-variable function as a one-variable function of x_k, $(k = 1, 2, ..., n)$. Here is an example.

Example 4.2.4. *Given positive real numbers $x_1, x_2, ..., x_n \in [a, b]$, find the maximum value of*

$$(x_1 - x_2)^2 + (x_1 - x_3)^2 + \cdots + (x_1 - x_n)^2 + (x_2 - x_3)^2 + \cdots + (x_{n-1} - x_n)^2.$$

(Mathematics and Youth Magazine)

SOLUTION. Denote the above expression by F. Notice that F, represented as a function of x_1 (we have already fixed other variables), is equal to

$$f(x_1) = (n-1)x_1^2 - 2\left(\sum_{i=2}^{n} x_i\right) x_1 + c$$

in which c is a constant. Clearly, f is a convex function ($f''(x) = 2(n-1) \geq 0$). According to the above lemma, we conclude that F attains the maximum value if and only if $x_i \in \{a, b\}$ for all $i \in \{1, 2, ..., n\}$. Suppose that k numbers x_i are equal to a and $(n-k)$ numbers x_i are equal to b. In this case, we have

$$F = n\left(\sum_{i=1}^{n} x_i^2\right) - \left(\sum_{i=1}^{n} x_i\right)^2 = nka^2 + n(n-k)b^2 - (ka + (n-k)b)^2 = k(n-k)(a-b)^2.$$

We conclude that

$$\max(F) = \begin{cases} m^2(a-b)^2 \text{ if } n = 2m, \ m \in \mathbb{N}, \\ m(m+1)(a-b)^2 \text{ if } n = 2m+1, \ m \in \mathbb{N}. \end{cases}$$

∇

Example 4.2.5. *Let $n \in \mathbb{N}$. Find the minimum value of the following expression*

$$f(x) = |1 + x| + |2 + x| + ... + |n + x|, \quad (x \in \mathbb{R}).$$

SOLUTION. Denote $I_1 = [-1, +\infty)$, $I_{n+1} = (-\infty, -n]$ and $I_k = [-k, -k+1]$ for each $k \in \{2, 3, ..., n\}$. If $x \in I_1$ then $f(x) = \sum_{i=1}^{n}(1 + x) \geq \sum_{i=1}^{n}(i - 1) = \frac{n(n-1)}{2} = f(-1)$. If $x \in I_n$ then $f(x) = \sum_{i=1}^{n}(-1 - x) \geq \sum_{i=1}^{n}(-i + n) = \frac{n(n-1)}{2} = f(-n)$.

Suppose $x \in I_k$ with $1 < k < n + 1$, then

$$f(x) = -\sum_{i=1}^{k-1}(i + x) + \sum_{i=k}^{n}(i + x)$$

is a linear function of x, therefore

$$\min_{x \in I_k} f(x) = \min\{f(-k), f(-k+1)\}.$$

This result, combined with the previous results, implies that

$$\min_{x\in\mathbb{R}} f(x) = \min\{f(-1), f(-2), ..., f(-n)\}.$$

After a simple calculation, we have

$$f(-k) = (1+2+...+(k-1)) + (1+2+...+(n-k)) = \frac{1}{2}(k^2+(n-k)^2+n),$$

which implies that

$$\min_{x\in\mathbb{R}} f(x) = \min_{1\le k\le n} \frac{k^2+(n-k)^2+n}{2} = \begin{cases} m(m+1) \text{ if } n = 2m\ (m\in\mathbb{N}). \\ (m+1)^2 \text{ if } n = 2m+1\ (m\in\mathbb{N}). \end{cases}$$

∇

Why does this approach make a great advantage? It helps us find the solutions immediately. It takes no time to try intermediate estimations, because everything we need to do is check the boundary values.

Sometimes, variables may be restricted not only in a certain interval but also by a mutual relationship. In this case, the following result is significant

Lemma 5. *Suppose that $f(x)$ is a real convex function defined on $[a,b] \in \mathbb{R}$ and $x_1, x_2, ..., x_n \in [a,b]$ such that $x_1 + x_2 + ... + x_n = s =$ constant $(na \le s \le nb)$. Consider the following expression*

$$F = f(x_1) + f(x_2) + ... + f(x_n).$$

F attains the maximum value if and only if at least $n-1$ elements of the sequence $(x_1, x_2, .., x_n)$ are equal to a or b.

SOLUTION. Notice that this lemma can be obtained directly from its case $n = 2$. In fact, it's sufficient to prove that: *if $x, y \in [a,b]$ and $2a \le x+y = s \le 2b$ then*

$$f(x)+f(y) \le \begin{cases} f(a)+f(s-a) \text{ if } s \le a+b. \\ f(b)+f(s-b) \text{ if } s \ge a+b. \end{cases}$$

Indeed, suppose that $s \le a+b$, then $s-a \le b$. Because $x \in [a, s-a]$, there exists a number $t \in [0,1]$ for which $x = ta+(1-t)(s-a)$. That gives $y = (1-t)a+t(s-a)$. By the definition of a convex function, we have $f(x) \le tf(a)+(1-t)f(s-a)$ and $f(y) \le (1-t)f(a)+tf(s-a)$. Adding up these results, we conclude

$$f(x)+f(y) \le f(a)+f(s-a).$$

In the case $s \geq a+b$, the lemma is proved similarly.

∇

Now look at some familiar problems that turn out to be simple by this theorem.

Example 4.2.6. *Let $a_1, a_2, ..., a_n$ be positive real numbers belonging to $[0,2]$ such that $a_1 + a_2 + ... + a_n = n$. Find the maximum value of*

$$S = a_1^2 + a_2^2 + ... + a_n^2.$$

SOLUTION. Applying the above lemma to the convex function $f(x) = x^2$, we get that S attains the maximum if and only if k numbers are equal to 2 and $n-k-1$ numbers are equal to 0. In this case, we have $S = 4k + (n-2k)^2$. Because $a_1, a_2, ..., a_n \in [0,2]$, we must have $0 \leq n - 2k \leq 2$.

If $n = 2m$ $(m \in \mathbb{N})$ then $n - 2k \in \{0, 2\}$. That implies $\max S = 4m = 2n$.
If $n = 2m+1$ $(m \in \mathbb{N})$ then $n - 2k = 1$. That implies $\max S = 4m + 1 = 2n + 1$.

∇

Example 4.2.7. *Suppose that $a, b, c \in [0,2]$ and $a + b + c = 5$. Prove that*

$$a^2 + b^2 + c^2 \leq 9.$$

SOLUTION. Suppose that $a \leq b \leq c$. According to lemma 5, we deduce that $a^2+b^2+c^2$ attains the maximum if and only if $a = 0$ or $b = c = 2$. The first case $a = 0$ is rejected because so, $4 \geq b + c = 5$ is a contradiction. In the second case, we have $a = 1$ and therefore $\max\{a^2 + b^2 + c^2\} = 1^2 + 2^2 + 2^2 = 9$.

∇

Example 4.2.8. *Let $a_1, a_2, ..., a_{2007}$ be real numbers in $[-1,1]$ such that $a_1 + a_2 + ... + a_{2007} = 0$. Prove that*

$$a_1^2 + a_2^2 + ... + a_{2007}^2 \leq 2006.$$

SOLUTION. Applying lemma 5, we deduce that the expression $a_1^2 + a_2^2 + ... + a_{2007}^2$ attains the maximum if and only if k numbers equal 1 and $n-k-1$ numbers equal -1. k must be 1003 and the last number must be 0, so we deduce that

$$a_1^2 + a_2^2 + ... + a_{2007}^2 \leq 2006.$$

∇

Example 4.2.9. *Let $x_1, x_2, ..., x_n$ be real numbers in the interval $[-1, 1]$ such that $x_1^3 + x_2^3 + ... + x_n^3 = 0$. Find the maximum value of $x_1 + x_2 + ... + x_n$.*

(Tran Nam Dung)

SOLUTION. We denote $a_i = x_i^3$ for all $i \in \{1, 2, ..., n\}$, then $a_1 + a_2 + ... + a_n = 0$. Notice that the function $f(x) = \sqrt[3]{x}$ is concave if $x \geq 0$ and convex if $x \leq 0$. It's easy to get (as in lemma 5)

$$f(x) + f(y) \leq \begin{cases} f(-1) + f(x+y+1) \text{ if } x, y \in [-1, 0]. \\ 2f\left(\dfrac{x+y}{2}\right) \text{ if } x, y \in [0, 1]. \end{cases} \tag{1}$$

Now suppose that there are two numbers x, y of the x_i's such that $x < 0 < y$ then

$$f(x) + f(y) \leq \begin{cases} f(-1) + f(x+y+1), \text{ if } x+y \leq 0. \\ f(0) + f(x+y), \text{ if } x+y \geq 0. \end{cases} \tag{2}$$

According to (1), we deduce that if $a_1, a_2, ..., a_k$ are all non-positive terms of the sequence $(a_1, a_2, ..., a_n)$, then

$$\sum_{i=1}^{k} f(a_i) \leq (k-1)f(1) + f\left(\sum_{i=1}^{n} a_i + k - 1\right),$$

which implies that we can change $k-1$ non-positive numbers to -1 to make the sum $\sum\limits_{i=1}^{k} f(a_i)$ bigger. Moreover, if $a_{k+1}, a_{k+2}, ..., a_n$ are non-negative then

$$\sum_{j=k+1}^{n} f(a_j) \leq (n-k)f\left(\frac{1}{n-k}\sum_{j=k+1}^{n} a_j\right).$$

That means we can replace all non-negative numbers with their arithmetic mean to make $\sum\limits_{i=k+1}^{n} f(a_i)$ bigger. However, we can make only $k-1$ non-positive numbers equal to -1; there is always one non-positive number left. Suppose that this number is a_k. Because $\sum\limits_{i=1}^{n} a_i = 0$, there is one non-negative number, say a_n. According to (2), because $a_k \leq 0 \leq a_n$, we can replace (a_k, a_n) with $(-1, a_k + a_n + 1)$ if $a_k + a_n \leq 0$ $(a_k + a_n + 1 \geq 0)$ and with $(0, a_k + a_n)$ if $a_k + a_n \geq 0$. After this step, the new sequence has all k non-negative elements equal to -1. Therefore

$$\sum_{i=1}^{n} f(a_i) \leq g(k) = kf(-1) + (n-k)f\left(\frac{k}{n-k}\right) = \sqrt[3]{k(n-k)^2} - k.$$

Notice that the derivative $g'(k)$ has only one root $k = \frac{n}{9}$, so we conclude that the maximum of the expression $\sum_{i=1}^{n} f(a_i)$ or $\sum_{i=1}^{n} x_i$ is

$$\max\left\{\sqrt[3]{\left[\frac{n}{9}\right]\cdot\left(n-\left[\frac{n}{9}\right]\right)^2}-\left[\frac{n}{9}\right], \sqrt[3]{\left(\left[\frac{n}{9}\right]-1\right)\cdot\left(n-\left[\frac{n}{9}\right]-1\right)^2}-\left[\frac{n}{9}\right]-1\right\}.$$

∇

Chapter 5

Abel Formula and Rearrangement Inequality

5.1 Abel formula

In the following pages, we will discuss an identity that closely relates to many problems in mathematics contests. In the field of inequality this identity has its best effect. It's called **Abel** formula.

Theorem 8 (Abel formula). *Suppose that $(x_1, x_2, ..., x_n)$ and $(y_1, y_2, ..., y_n)$ are two sequences of real numbers. Denote $c_k = y_1 + y_2 + ... + y_k$ $(k = 1, 2, ..., n)$, then*

$$x_1y_1 + x_2y_2 + ... + x_ny_n = (x_1 - x_2)c_1 + (x_2 - x_3)c_2 + ... + (x_{n-1} - x_n)c_{n-1} + x_nc_n.$$

PROOF. We certainly have

$$(x_1 - x_2)c_1 + (x_2 - x_3)c_2 + ... + (x_{n-1} - x_n)c_{n-1} + x_nc_n$$

$$= c_1x_1 + (c_2 - c_1)x_2 + ... + (c_n - c_{n-1})x_n = x_1y_1 + x_2y_2 + ... + x_ny_n.$$

∇

From this theorem, the following result can be obtained directly

Example 5.1.1 (Abel inequality). *Let $x_1, x_2, ..., x_n$ and $y_1 \geq y_2 \geq ... \geq y_n \geq 0$ be real numbers. For each $k \in \{1, 2, ..., n\}$, we denote $S_k = \sum\limits_{i=1}^{k} x_i$. Suppose that $M = \max\{S_1, S_2, ..., S_n\}$ and $m = \min\{S_1, S_2, ..., S_n\}$, then*

$$my_1 \leq x_1y_1 + x_2y_2 + ... + x_ny_n \leq My_1.$$

SOLUTION. Since both two parts of the inequality can be proved similarly, we only need to show the solution to the left inequality. Let $y_{n+1} = 0$. By **Abel** formula

$$\sum_{i=1}^{n} x_i y_i = \sum_{i=1}^{n} (y_i - y_{i+1}) S_i \geq \sum_{i=1}^{n} m(y_i - y_{i+1}) = m y_1.$$

∇

Inequalities solved by **Abel** formula often appear in sophisticated conditions that makes them difficult to solve by other methods. Here are some examples.

Example 5.1.2. *Let $a_1, a_2, .., a_n$ and $b_1 \geq b_2 \geq ... \geq b_n \geq 0$ be positive real numbers such that $a_1 a_2 ... a_k \geq b_1 b_2 ... b_k$ $\forall k \in \{1, 2, ..., n\}$. Prove the following inequality*

$$a_1 + a_2 + ... + a_n \geq b_1 + b_2 + ... + b_n.$$

SOLUTION. By **Abel** formula, we deduce that

$$\sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i = \sum_{i=1}^{n} b_i \left(\frac{a_i}{b_i} - 1 \right)$$

$$= (b_1 - b_2)\left(\frac{a_1}{b_1} - 1\right) + (b_2 - b_3)\left(\frac{a_1}{b_1} + \frac{a_2}{b_2} - 2\right) + ...$$

$$+ (b_{n-1} - b_n)\left(\sum_{i=1}^{n-1} \frac{a_i}{b_i} - n + 1\right) + b_n \left(\sum_{i=1}^{n} \frac{a_i}{b_i} - n\right) \geq 0,$$

because **AM-GM** inequality yields that for all $k \in \{1, 2, ..., n\}$

$$\frac{a_1}{b_1} + \frac{a_2}{b_2} + \ldots + \frac{a_k}{b_k} \geq k \sqrt[k]{\frac{a_1 a_2 ... a_k}{b_1 b_2 ... b_k}} \geq k.$$

∇

Example 5.1.3. *Let $x_1, x_2, .., x_n$ be positive real numbers such that*

$$x_1 + x_2 + ... + x_k \geq \sqrt{k} \ \forall k \in \{1, 2, ..., n\}.$$

Prove the following inequality

$$x_1^2 + x_2^2 + ... + x_n^2 \geq \frac{1}{4}\left(1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}\right).$$

(USA MO 1994)

SOLUTION. WLOG, assume that $x_1 \geq x_2 \geq ... \geq x_n$. For each $k \in \{1, 2, ..., n\}$, let $b_k = \dfrac{1}{\sqrt{k}}$. We will first prove that

$$2\sum_{i=1}^{n} x_i^2 \geq \sum_{i=1}^{n} x_i b_i.$$

and then, $2\sum_{i=1}^{n} x_i b_i \geq \sum_{i=1}^{n} b_i^2$.

By **Abel** formula, we have

$$\sum_{i=1}^{n} x_i(2x_i - b_i) = (x_1 - x_2)(2x_1 - b_1) + (x_2 - x_3)(2x_1 + 2x_2 - b_1 - b_2) + \ldots$$

$$+ (x_{n-1} - x_n)\left(2\sum_{i=1}^{n-1} x_i - \sum_{i=1}^{n-1} b_i\right) + x_n\left(2\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} b_i\right)$$

Because $x_k \geq x_{k+1}$ $\forall k \in \{1, 2, ..., n\}$, so we only need to prove that $2\sum_{i=1}^{k} x_i \geq \sum_{i=1}^{k} b_i$. By hypothesis, it's enough to prove that

$$\sum_{i=1}^{k} \frac{1}{\sqrt{i}} \leq 2\sqrt{k}.$$

However, this last inequality is clearly true because

$$\sum_{i=1}^{k} \frac{1}{\sqrt{i}} \leq \sum_{i=1}^{k} \frac{2}{\sqrt{i} + \sqrt{i-1}} = 2\sum_{i=1}^{k} \left(\sqrt{i} - \sqrt{i-1}\right) = 2\sqrt{k}.$$

∇

Also by Abe formula,

$$\sum_{i=1}^{n} b_i(2x_i - b_i) = (b_1 - b_2)(2x_1 - b_1) + \ldots + (b_{n-1} - b_n)\left(2\sum_{i=1}^{n-1} x_i - \sum_{i=1}^{n-1} b_i\right) +$$

$$+ b_n\left(2\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} b_i\right).$$

$b_n \geq b_{k+1}$, $(\forall)$ $k \in \{1, 2, \ldots, n\}$, so all terms are positive.

Example 5.1.4. *Let $a_1, a_2, ..., a_n$ and $b_1, b_2, ..., b_n$ be real numbers such that*

$$a_1 \geq \frac{a_1 + a_2}{2} \geq \ldots \geq \frac{a_1 + a_2 + \ldots + a_n}{n},$$
$$b_1 \geq \frac{b_1 + b_2}{2} \geq \ldots \geq \frac{b_1 + b_2 + \ldots + b_n}{n}.$$

Prove the following inequality

$$a_1 b_1 + a_2 b_2 + \ldots + a_n b_n \geq \frac{1}{n}(a_1 + a_2 + \ldots + a_n)(b_1 + b_2 + \ldots + b_n).$$

(Improved Chebyshev inequality)

SOLUTION. For each $k \in \{1, 2, ..., n\}$, we denote $S_k = a_1 + a_2 + ... + a_k$ and $b_{n+1} = 0$. By **Abel** formula, we have

$$\sum_{i=1}^{n} a_i b_i = \sum_{i=1}^{n} (b_i - b_{i+1}) S_i = \sum_{i=1}^{n} i(b_i - b_{i+1}) \left(\frac{S_i}{i}\right).$$

According to **Abel** formula again, we have

$$\sum_{i=1}^{n} a_i b_i = \left(S_1 - \frac{S_2}{2}\right)(b_1 - b_2) + \left(\frac{S_2}{2} - \frac{S_3}{3}\right)(b_1 + b_2 - 2b_3) + ...$$
$$+ \left(\frac{S_{n-1}}{n-1} - \frac{S_n}{n}\right)\left(\sum_{i=1}^{n-1} b_i - (n-1)b_n\right) + \frac{1}{n}\left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} b_i\right).$$

By hypothesis, we have $\frac{S_1}{1} \geq \frac{S_2}{2} \geq ... \geq \frac{S_n}{n}$, so it's enough to prove that

$$\sum_{i=1}^{k} b_i \geq k b_{k+1} \; \forall k \in \{1, 2, ..., n-1\}.$$

This one comes directly from the hypothesis

$$\frac{1}{k}\sum_{i=1}^{k} b_i \geq \frac{1}{k+1}\sum_{i=1}^{k+1} b_i.$$

∇

Example 5.1.5. *Let $x_1, x_2, ..., x_n$ be real numbers such that $x_1 \geq x_2 \geq ... \geq x_n \geq x_{n+1} = 0$. Prove the following inequality*

$$\sqrt{x_1 + x_2 + ... + x_n} \leq \sum_{i=1}^{n} \sqrt{i}(\sqrt{x_i} - \sqrt{x_{i+1}}).$$

(Romania MO and Singapore MO)

SOLUTION. Denote $c_i = \sqrt{i} - \sqrt{i-1}$ and $a_i = \sqrt{x_i}$. The inequality becomes

$$(a_1c_1 + a_2c_2 + ... + a_nc_n)^2 \geq a_1^2 + a_2^2 + ... + a_n^2.$$

Suppose that $b_1, b_2, ..., b_n$ are positive real numbers satisfying $\sum\limits_{i=1}^{n} b_i^2 = 1$ and $\left(\sum\limits_{i=1}^{n} a_i^2\right)\left(\sum\limits_{i=1}^{n} b_i^2\right) = \left(\sum\limits_{i=1}^{n} a_ib_i\right)^2$ (the sequences $(a_1, a_2, ..., a_n)$ and $(b_1, b_2, ..., b_n)$ are proportional). We need to prove that

$$a_1c_1 + a_2c_2 + ... + a_nc_n \geq a_1b_1 + a_2b_2 + ... + a_nb_n.$$

By **Abel** formula, the above inequality can be changed into

$$\sum_{i=1}^{n} a_i(c_i - b_i) \geq 0 \Leftrightarrow (a_1 - a_2)(c_1 - b_1) + (a_2 - a_3)(c_1 + c_2 - b_1 - b_2) + \ldots.$$

$$+ (a_{n-1} - a_n)\left(\sum_{i=1}^{n-1} a_i - \sum_{i=1}^{n-1} b_i\right) + a_n\left(\sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i\right) \geq 0.$$

which is true because for all $k = 1, 2, ..., n$.

$$\sum_{i=1}^{k} c_i - \sum_{i=1}^{k} b_i = \sqrt{k} - \sum_{i=1}^{k} b_i \geq \sqrt{k} - \sqrt{k\left(\sum_{i=1}^{k} b_i^2\right)} \geq 0.$$

∇

Example 5.1.6. *Let $a_1, a_2, ..., a_n$ and $b_1 \leq b_2 \leq ... \leq b_n$ be real numbers such that $a_1^2 + a_2^2 + ... + a_k^2 \leq b_1^2 + b_2^2 + ... + b_k^2$ $\forall k \in \{1, 2, ..., n\}$. Prove that*

$$a_1 + a_2 + ... + a_n \leq b_1 + b_2 + ... + b_n.$$

SOLUTION. We prove this problem by induction. Case $n = 1$ is obvious. Suppose that the problem has been proved for n numbers already. We will prove it for $n + 1$ numbers. Indeed, by **Cauchy-Schwarz**, we deduce that

$$(a_1^2 + a_2^2 + ... + a_{n+1}^2)(b_1^2 + b_2^2 + ... + b_{n+1}^2) \geq (a_1b_1 + a_2b_2 + ... + a_{n+1}b_{n+1})^2.$$

By hypothesis that $\sum\limits_{i=1}^{n+1} a_i^2 \leq \sum\limits_{i=1}^{n+1} b_i^2$, so $\sum\limits_{i=1}^{n+1} b_i^2 \geq \sum\limits_{i=1}^{n+1} a_ib_i$. According to **Abel** formula,

$$0 \leq \sum_{i=1}^{n+1} b_i(b_i - a_i) = (b_1 - b_2)(b_1 - a_1) + (b_2 - b_3)(b_1 + b_2 - a_1 - a_2) + \ldots$$

$$+ (b_n - b_{n+1})\left(\sum_{i=1}^{n} b_i - \sum_{i=1}^{n} a_i\right) + b_{n+1}\left(\sum_{i=1}^{n+1} b_i - \sum_{i=1}^{n+1} a_i\right).$$

In the sum above, every term except the last one is non-positive (because for all $k \in \{1, 2, ..., n\}$ we have $b_k \leq b_{k+1}$ and $\sum\limits_{i=1}^{k} b_i \geq \sum\limits_{i=1}^{k} a_i$, by inductive hypothesis). So we must have that

$$b_{n+1}\left(\sum_{i=1}^{n+1} b_i - \sum_{i=1}^{n+1} a_i\right) \geq 0 \Leftrightarrow \sum_{i=1}^{n+1} b_i \geq \sum_{i=1}^{n+1} a_i.$$

Comment. The following stronger result, proposed by Le Huu Dien Khue, can be proved directly (without induction) by **Abel** formula

★ *Let $a_1, a_2, ..., a_n$ and $b_1 \le b_2 \le ... \le b_n$ be real numbers such that $a_1^2 + a_2^2 + ... + a_k^2 \le b_1^2 + b_2^2 + ... + b_k^2$ $\forall k \in \{1, 2, ..., n\}$. Prove that*

$$b_1 + b_2 + ... + b_n \ge \frac{a_1^2}{b_1} + \frac{a_2^2}{b_2} + ... + \frac{a_n^2}{b_n}.$$

∇

Example 5.1.7. *Let $-1 < x_1 < x_2 < ... < x_n < 1$ and $y_1 < y_2 < ... < y_n$ be real numbers such that $x_1 + x_2 + ... + x_n = x_1^{13} + x_2^{13} + ... + x_n^{13}$. Prove that*

$$x_1^{13}y_1 + x_2^{13}y_2 + ... + x_n^{13}y_n < x_1y_1 + x_2y_2 + ... + x_ny_n.$$

(Russia MO 2000)

SOLUTION. According to **Abel** formula, we note that

$$\sum_{i=1}^{n} y_i\left(x_i^{13} - x_i\right) = (y_1 - y_2)\left(x_1^{13} - x_1\right) + (y_2 - y_3)\left(x_1^{13} + x_2^{13} - x_1 - x_2\right) + ...$$
$$+ (y_{n-1} - y_n)\left(\sum_{i=1}^{n-1} x_i^{13} - \sum_{i=1}^{n-1} x_i\right) + y_n\left(\sum_{i=1}^{n} x_i^{13} - \sum_{i=1}^{n} x_i\right).$$

Because $y_k \le y_{k+1}$ $\forall k \in \{1, 2, ..., n-1\}$, we only need to prove that

$$\sum_{i=1}^{k} x_i^{13} \ge \sum_{i=1}^{k} x_i \Leftrightarrow \sum_{i=1}^{k} x_i\left(x_i^{12} - 1\right) \ge 0.$$

Applying **Abel** formula again, we have

$$\sum_{i=1}^{k} x_i\left(x_i^{12} - 1\right) = (x_1 - x_2)\left(x_1^{12} - 1\right) + (x_2 - x_3)\left(x_1^{12} + x_2^{12} - 2\right) + ...$$
$$+ (x_{k-1} - x_k)\left(\sum_{i=1}^{k-1} x_i^{12} - k + 1\right) + x_k\left(\sum_{i=1}^{k} x_i^{12} - k\right).$$

Notice that $x_i \in [-1, 1]$, $\forall i \in \{1, 2, ..., n\}$ so $\sum\limits_{i=1}^{j} x_i^{12} \le j$ $\forall j \in \{1, 2, ..., k\}$. Moreover, because $x_1 \le x_2 \le ... \le x_k$, every term in the above sum except the last term is non-negative. If $x_k \le 0$, we are done. Otherwise, suppose that $x_k \ge 0$, then $x_i \ge 0$ $\forall i \ge k+1$. This implies (by hypothesis)

$$\sum_{i=k+1}^{n} x_i^{13} \le \sum_{i=k+1}^{n} x_i \Rightarrow \sum_{i=1}^{k} x_i^{13} \ge \sum_{i=1}^{k} x_i.$$

∇

Problems solved by **Abel** formula as above are a bit unusual. The strength of **Abel** formula is shown in inequalities with sequences, where other methods fail. **Abel** formula is also significant in the proof of a very important inequality that will be discussed now, *rearrangement inequality.*

5.2 Rearrangement Inequality

Theorem 9 (Rearrangement Inequality). *Let $(a_1, a_2, ..., a_n)$ and $(b_1, b_2, ..., b_n)$ be two increasing sequences of real numbers. Suppose that $(i_1, i_2, ..., i_n)$ is an arbitrary permutation of $(1, 2, ..., n)$, then*

$$a_1b_1 + a_2b_2 + ... + a_nb_n \geq a_1b_{i_1} + a_2b_{i_2} + ... + a_nb_{i_n}$$

If the sequence $(a_1, a_2, ..., a_n)$ is increasing but the sequence $(b_1, b_2, ..., b_n)$ is decreasing then the sign of the above inequality is reversed.

PROOF. Notice that $a_1 \leq a_2 \leq ... \leq a_n$ and $b_1 \leq b_2 \leq ... \leq b_n$, so according to **Abel** formula,

$$\begin{aligned}\sum_{k=1}^{n} a_kb_k - \sum_{k=1}^{n} a_kb_{i_k} &= \sum_{k=1}^{n} a_k(b_k - b_{i_k}) \\ &= (a_1 - a_2)(b_1 - b_{i_1}) + (a_2 - a_b)(b_1 + b_2 - b_{i_1} - b_{i_2}) + ... \\ &\quad (a_{n-1} - a_n)\left(\sum_{k=1}^{n-1} b_k - \sum_{k=1}^{n-1} b_{i_k}\right) + a_n\left(\sum_{k=1}^{n} b_k - \sum_{k=1}^{n} b_{i_k}\right) \geq 0,\end{aligned}$$

because for all $k \in \{1, 2, ..., n\}$, we have $\sum\limits_{j=1}^{k} b_j \leq \sum\limits_{j=1}^{k} b_{i_j}$.

The theorem is proved similarly in the case with $(a_1, a_2, ..., a_n)$ increasing but $(b_1, b_2, ..., b_n)$ is decreasing.

∇

In practice, **Rearrangement** inequality is strong. It can help prove **AM-GM** inequality in a single way.

Example 5.2.1. *Let $a_1, a_2, .., a_n$ be positive real numbers. Prove that*

$$a_1 + a_2 + ... + a_n \geq n\sqrt[n]{a_1a_2...a_n}.$$

SOLUTION. WLOG, assume that $a_1a_2...a_n = 1$ (normalization). Let $a_1 = \frac{x_1}{x_2}, a_2 = \frac{x_2}{x_3}, ..., a_{n-1} = \frac{x_{n-1}}{x_n}$, $x_1, x_2, ..., x_n > 0$, then $a_n = \frac{x_n}{x_1}$. The problem becomes

$$\frac{x_1}{x_2} + \frac{x_2}{x_3} + ... + \frac{x_{n-1}}{x_n} + \frac{x_n}{x_1} \geq n.$$

Notice that if the sequence $(x_1, x_2, ..., x_n)$ is increasing then the sequence $\left(\frac{1}{x_1}, \frac{1}{x_2}, ..., \frac{1}{x_n}\right)$ is decreasing. By **Rearrangement** inequality, we conclude that

$$\sum_{i=1}^{n} \frac{x_i}{x_{i+1}} = \sum_{i=1}^{n} x_i \cdot \frac{1}{x_{i+1}} \geq \sum_{i=1}^{n} x_i \cdot \frac{1}{x_i} = n.$$

∇

For cyclic inequalities, **Rearrangement** inequality seems to be very effective. Sometimes it's not easy to realize the use of **Rearrangement** inequality because it is hidden after the normal order of variables is changed into chaos. Being aware of **Rearrangement** inequality in a problem requires a bit more intuitive ability then for other inequalities. The following problems will, hopefully, enhance that ability.

Example 5.2.2. *Suppose that a, b, c are the side-lengths of a triangle. Prove that*

$$a^2b(a-b) + b^2c(b-c) + c^2a(c-a) \geq 0.$$

(IMO 1984, A3)

Solution. Because a, b, c are the side-lengths of a triangle, $a \geq b$ implies $a^2 + bc \geq b^2 + ca$. By this property, we deduce that if $a \geq b \geq c$ then $a^2 + bc \geq b^2 + ca \geq c^2 + ab$; also, $\frac{1}{a} \leq \frac{1}{b} \leq \frac{1}{c}$. According to **Rearrangement** inequality, we conclude that

$$\sum_{cyc} \frac{a^2+bc}{a} \leq \sum_{cyc} \frac{a^2+bc}{c} \Rightarrow \sum_{cyc} \frac{bc}{a} \leq \sum_{cyc} \frac{a^2}{c} \Rightarrow \sum_{cyc} a^2b^2 \leq \sum_{cyc} a^3b,$$

which is equivalent to the desired result. Equality holds for $a = b = c$.

∇

Example 5.2.3. *Let a, b, c be positive real numbers. Prove that*

$$\frac{a^2+bc}{b+c} + \frac{b^2+ca}{c+a} + \frac{c^2+ab}{a+b} \geq a+b+c.$$

Solution. Applying **Rearrangement** inequality for the sequences (a^2, b^2, c^2) and $\left(\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}\right)$ (if $a \geq b \geq$ c, then these are both increasing), we get that

$$\sum_{cyc} \frac{a^2}{b+c} \geq \sum_{cyc} \frac{b^2}{b+c},$$

which implies that

$$\sum_{cyc} \frac{a^2+bc}{b+c} \geq \sum_{cyc} \frac{b^2}{b+c} + \sum_{cyc} \frac{bc}{b+c} = \sum_{cyc} a.$$

∇

Example 5.2.4. *Let a, b, c be positive real numbers. Prove that*

$$\frac{a+b}{a+c}+\frac{a+c}{b+c}+\frac{b+c}{a+b} \le \frac{a}{b}+\frac{b}{c}+\frac{c}{a}.$$

(Mathlinks Contest)

SOLUTION. The inequality can be rewritten to

$$\sum_{cyc}\left(\frac{a}{b}-\frac{a}{b+c}\right) \ge \sum_{cyc}\frac{a}{a+c} \Leftrightarrow \sum_{cyc}\frac{ac}{b(b+c)} \ge \sum_{cyc}\frac{a}{a+c}.$$

Consider the expressions $P=\sum_{cyc}\frac{ac}{b(b+c)}$ and $Q=\sum_{cyc}\frac{bc}{a(b+c)}$. By **Rearrangement inequality**, we deduce that

$$Q=\sum_{cyc}\frac{bc}{a}\cdot\frac{1}{b+c} \le \sum_{cyc}\frac{ac}{b}\cdot\frac{1}{b+c}=P.$$

Moreover, by **Cauchy-Schwarz** inequality, we can write

$$PQ \ge \left(\sum_{cyc}\frac{a}{a+c}\right)^2$$

and therefore (because $P \ge Q$) we have $P \ge \sum_{cyc}\frac{a}{a+c}$. The proof is finished and the equality occurs if and only if $a=b=c$.

Comment. This inequality can be proved by another nice approach. Indeed, notice that for all positive real numbers $a, b, c > 0$, we have

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-3=\frac{(a-b)^2}{ab}+\frac{(a-c)(b-c)}{ac}.$$

WLOG, assume that $c=\min(a,b,c)$. Rewrite the inequality to

$$\left[\frac{1}{ab}-\frac{1}{(a+c)(b+c)}\right](a-b)^2+\left[\frac{1}{ac}-\frac{1}{(a+c)(a+b)}\right](a-c)(b-c) \ge 0$$

which is obvious because $c=\min(a,b,c)$.

∇

Example 5.2.5. *Let a, b, c be positive real numbers. Prove that*

$$\frac{a+b}{b+c}+\frac{b+c}{c+a}+\frac{c+a}{a+b} \le \frac{(a+b+c)^2}{ab+bc+ca}.$$

SOLUTION. The inequality is equivalent to

$$\sum_{cyc} \frac{(a+b)\,(a(b+c)+bc)}{b+c} \le (a+b+c)^2$$

$$\Leftrightarrow \sum_{cyc} a(a+b) + \sum_{cyc} \frac{bc(a+b)}{b+c} \le (a+b+c)^2$$

$$\Leftrightarrow \sum_{cyc} \left(\frac{bc}{b+c}\right)(a+b) \le ab+bc+ca.$$

This last inequality is true by **Rearrangement** inequality because if $x \ge y \ge z$ then

$$x+y \ge x+z \ge y+z \quad ; \quad \frac{xy}{x+y} \ge \frac{xz}{x+z} \ge \frac{yz}{y+z} \quad ;$$

Equality holds for $a=b=c$.

∇

Example 5.2.6. *Let a,b,c,d be non-negative real numbers such that $a+b+c+d=4$. Prove that*

$$a^2bc + b^2cd + c^2da + d^2ab \le 4.$$

(Song Yoon Kim)

SOLUTION. Suppose that (x,y,z,t) is a permutation of (a,b,c,d) such that $x \ge y \ge z \ge t$, then $xyz \ge xyt \ge xzt \ge yzt$. By **Rearrangement** inequality, we deduce that

$$x \cdot xyz + y \cdot xyt + z \cdot xzt + t \cdot yzt \ge a^2bc + b^2cd + c^2da + d^2ab.$$

According to **AM-GM** inequality, we also have

$$\begin{aligned} x \cdot xyz + y \cdot xyt + z \cdot xzt + t \cdot yzt &= (xy+zt)(xz+yt) \\ &\le \frac{1}{4}(xy+xz+yt+zt)^2 \le 4, \end{aligned}$$

because $xy+yz+zt+tx = (x+z)(y+t) \le \frac{1}{4}(x+y+z+t)^2 = 4$. Equality holds for $a=b=c=1$ or $a=2, b=c=1, c=0$ up to permutation.

∇

Example 5.2.7. *Let a,b,c,d be positive real numbers. Prove that*

$$\left(\frac{a}{a+b+c}\right)^2 + \left(\frac{b}{b+c+d}\right)^2 + \left(\frac{c}{c+d+a}\right)^2 + \left(\frac{d}{d+a+b}\right)^2 \ge \frac{4}{9}.$$

(Pham Kim Hung)

SOLUTION. WLOG, we may assume that $a+b+c+d=1$. Suppose that (x,y,z,t) is a permutation of (a,b,c,d) such that $x \geq y \geq z \geq t$, then

$$\frac{1}{x+y+z} \leq \frac{1}{x+y+t} \leq \frac{1}{x+z+t} \leq \frac{1}{y+z+t}.$$

By **Rearrangement** inequality, we deduce that

$$\sum_{cyc}\left(\frac{a}{a+b+c}\right)^2 \geq \frac{x^2}{(x+y+z)^2}+\frac{y^2}{(x+y+t)^2}+\frac{z^2}{(x+z+t)^2}+\frac{t^2}{(y+z+t)^2}$$

$$=\frac{x^2}{(1-t)^2}+\frac{y^2}{(1-z)^2}+\frac{z^2}{(1-y)^2}+\frac{t^2}{(1-x)^2}.$$

Denote $m=x+t$, $n=xt$ and $s=\dfrac{x^2}{(1-t)^2}+\dfrac{t^2}{(1-x)^2}$. Certainly, we only need to consider the case $s \leq \dfrac{1}{2}$. If $m=1$ then $y=z=0$ and the result is obvious because

$$\frac{x^2}{(1-t)^2}+\frac{y^2}{(1-z)^2}+\frac{z^2}{(1-y)^2}+\frac{t^2}{(1-x)^2}=\frac{x^2}{x^2}+\frac{t^2}{t^2}=2.$$

Otherwise, we have $m<1$ and $s \leq \dfrac{1}{2}$. After a short computation, we have

$$n^2(2-s)-2n(m-1)(2m-1-s)+(m-1)^2(m^2-s)=0.$$

This identity says that the function $f(\alpha)=\alpha^2(2-s)-2\alpha(m-1)(2m-1-s)+(m-1)^2(m^2-s)$ has at least one root $\alpha=n$. That implies

$$\Delta'_f=(m-1)^2(2m-1-s)^2-(2-s)(m-1)^2(m^2-s) \geq 0,$$

or equivalently

$$s \geq \frac{-2m^2+4m-1}{(2-m)^2}.$$

Similarly, we denote $p=y+z$ and $t=\dfrac{y^2}{(1-z)^2}+\dfrac{z^2}{(1-y)^2}$. The inequality is obvious if $t \geq \dfrac{1}{2}$ or $p=1$. Otherwise,

$$t \geq \frac{-2p^2+4p-1}{(2-p)^2}=\frac{1-2m^2}{(m+1)^2}.$$

It remains to prove that

$$\frac{-2m^2+4m-1}{(2-m)^2}+\frac{1-2m^2}{(m+1)^2} \geq \frac{4}{9} \Leftrightarrow \frac{(2m-1)^2(11+10m-10m^2)}{(2-m)^2(m+1)^2} \geq 0,$$

which is clearly true. Equality holds for $a=b=c=d$.

∇

According to hypothesis $xy + yz + zx = 1$, so we will figure out a number l such that $2l = \sqrt{2(k-l)}$. A simple calculation reveals $l = \dfrac{-1+\sqrt{1+8k}}{4}$, and therefore the final result

$$k(x^2 + y^2) + z^2 \geq \frac{-1+\sqrt{1+8k}}{2}.$$

Comment. The following more general problem can be solved by the same method.

★ *Suppose that x, y, z are positive real numbers verifying $xy + yz + zx = 1$ and k, l are two positive real constants. The minimum value of the expression*

$$kx^2 + ly^2 + z^2$$

is $2t_0$, where t_0 is the unique positive real root of the equation

$$2t^3 + (k + l + 1)t - kl = 0.$$

$$\nabla$$

By the same method, we will solve some other problems regarding *intermediate variables.*

Example 6.1.2. *Let x, y, z, t be real numbers satisfying $xy + yz + zt + tx = 1$. Find the minimum of the expression*

$$5x^2 + 4y^2 + 5z^2 + t^2.$$

SOLUTION. We choose a positive number $l < 5$ and apply **AM-GM** inequality

$$\begin{aligned}
lx^2 + 2y^2 &\geq 2\sqrt{2l}xy, \\
2y^2 + lz^2 &\geq 2\sqrt{2l}yz, \\
(5-l)z^2 + 1/2t^2 &\geq \sqrt{2(5-l)}zt, \\
1/2t^2 + (5-l)x^2 &\geq \sqrt{2(5-l)}tx.
\end{aligned}$$

Summing up these results, we conclude that

$$5x^2 + 4y^2 + 5z^2 + t^2 \geq 2\sqrt{2l}(xy + tz) + \sqrt{2(5-l)}(zt + tx).$$

The condition $xy + yz + zt + tx = 1$ suggests us to choose a number l $(0 \leq l \leq 5)$ such that $2\sqrt{2l} = \sqrt{2(5-l)}$. A simple calculation yields $l = 1$, thus, the minimum of $5x^2 + 4y^2 + 5z^2 + t^2$ is $2\sqrt{2}$.

Comment. The following general problem can be solved by the same method

★ *Let x, y, z, t be arbitrary real numbers. Prove that*

$$x^2 + ky^2 + z^2 + lt^2 \geq \sqrt{\frac{2kl}{k+l}}(xy + yz + zx + tx).$$

∇

Example 6.1.3. *Let x, y, z be positive real numbers with sum 3. Find the minimum of the expression $x^2 + y^2 + z^3$.*

(Pham Kim Hung)

SOLUTION. Let a and b be positive real numbers. By **AM-GM** inequality, we have

$$x^2 + a^2 \geq 2ax,$$
$$y^2 + a^2 \geq 2ay,$$
$$z^3 + b^3 + b^3 \geq 3b^2 z.$$

Combining these results yields that $x^2 + y^2 + z^3 + 2(a^2 + b^3) \geq 2a(x + y) + 3b^2 z$, with equality for $x = y = a$ and $z = b$. In this case, we could have $2a + b = x + y + z = 3$ $(\star)$. Moreover, in order for $2a(x + y) + 3b^2 z$ to be represented as $x + y + z$, we must have $2a = 3b^2$ $(\star\star)$. According to $(\star)$ and $(\star\star)$, we easily find out

$$b = \frac{-1 + \sqrt{37}}{6}, \quad a = \frac{3 - b}{2} = \frac{19 - \sqrt{37}}{12},$$

therefore the minimum of $x^2 + y^2 + z^3$ is $6a - (2a^2 + b^3)$ where a, b are determined as above. The proof is completed. ■

∇

Generally, to handle difficult problems by this method, we need to construct a lot of equations then solve them. This work (solving systems of equations) can be complicated but unavoidable. The following example presents such a severe trial.

Example 6.1.4. *Let a, b, c be three positive constants and x, y, z three positive variables such that $ax + by + cz = xyz$. Prove that if there exists a unique positive number d such that*

$$\frac{2}{d} = \frac{1}{a+d} + \frac{1}{b+d} + \frac{1}{c+d},$$

then the minimum of $x + y + z$ is

$$\sqrt{d(d+a)(d+b)(d+c)}.$$

(Nguyen Quoc Khanh, VMEO 2006)

SOLUTION. To avoid the complicated condition $ax + by + cz = xyz$, we will find the minimum of the following homogeneous expression

$$\frac{(ax+by+cz)(x+y+z)^2}{xyz}.$$

Certainly, if the minimum of the above expression is equal to C then the minimum value of $x+y+z$ is equal to C, too.

Assume that m, n, p, m_1, n_1, p_1 are arbitrary positive real numbers such that $m+n+p = am_1 + bn_1 + cp_1 = 1$. By the weighted AM-GM inequality, we have

$$x+y+z = m\left(\frac{x}{m}\right) + n\left(\frac{y}{n}\right) + p\left(\frac{z}{p}\right) \geq \frac{x^m y^n z^p}{m^m n^n p^p},$$

$$ax+by+cz = am_1\left(\frac{x}{m_1}\right) + bn_1\left(\frac{y}{n_1}\right) + cp_1\left(\frac{z}{p_1}\right) \geq \frac{x^{am_1} y^{bn_1} z^{cp_1}}{m_1^{am_1} n_1^{bn_1} p_1^{cp_1}},$$

$$\Rightarrow (ax+by+cz)(x+y+z)^2 \geq \frac{x^{am_1+2m} y^{bn_1+2n} z^{2p_1+2p}}{m^{2m} n^{2n} p^{2p} m_1^{am_1} n_1^{bn_1} p_1^{cp_1}},$$

with equality for $\frac{x}{m} = \frac{y}{n} = \frac{z}{p}, \frac{x}{m_1} = \frac{y}{n_1} = \frac{z}{p_1}$. Moreover, we also need the condition $am_1 + 2m = bn_1 + 2n = cp_1 + 2p = 1$. Denote $k = \frac{2m}{m_1} = \frac{2n}{n_1} = \frac{2p}{p_1}$, then

$$2am + 2bn + 2cp = k \quad \& \quad 2m\left(\frac{a}{k}+1\right) = 2n\left(\frac{b}{k}+1\right) = 2p\left(\frac{c}{k}+1\right) = 1.$$

These conditions combined yield that

$$\frac{ak}{a+k} + \frac{bk}{b+k} + \frac{ck}{c+k} = k \Leftrightarrow \frac{1}{a+k} + \frac{1}{b+k} + \frac{1}{c+k} = \frac{2}{k}.$$

Because d is unique, we must have $k = d$. After a simple calculation, we get that

$$m^{2m} n^{2n} p^{2p} m_1^{am_1} n_1^{bn_1} p_1^{cp_1} = d^{-1}(d+a)^{-1}(d+b)^{-1}(d+c)^{-1},$$

and the conclusion follows.

Comment. This inequality is generalized from the following problem in the **Vietnam TST** 2001 proposed by Tran Nam Dung

★ *Let a, b, c be positive real numbers such that $12 \geq 21ab + 2bc + 8ca$. Prove that*

$$\frac{1}{a} + \frac{2}{b} + \frac{3}{c} \geq \frac{15}{2}.$$

∇

In some situations, solving systems of equations to find intermediate variables is not at all *real computing*. Sometimes it completely depends on your own intuition, because solving these equations to find roots can be impossible. But you can guess these roots! That's what I want emphasize in the next example.

Example 6.1.5. *Let $x_1, x_2, ..., x_n$ be positive real numbers. Prove that*

$$x_1 + \sqrt{x_1x_2} + ... + \sqrt[n]{x_1x_2...x_n} \le e(x_1 + x_2 + ... + x_n).$$

SOLUTION. Suppose that $a_1, a_2, ..., a_n$ are positive real numbers. According to AM-GM inequality, we have

$$\sqrt[k]{(a_1x_1)\cdot(a_2x_2)\cdots(a_kx_k)} \le \frac{a_1x_1 + a_2x_2 + ... + a_kx_k}{k}.$$

$$\Rightarrow \sqrt[k]{x_1x_2...x_k} \le \frac{1}{k}\sum_{i=1}^{k} x_i \cdot \frac{a_i}{\sqrt[k]{a_1a_2....a_k}}.$$

Constructing similar result for all $k \in \{1, 2, ..., n\}$, we conclude that

$$\sum_{k=1}^{n} \sqrt[k]{x_1x_2...x_k} \le \sum_{k=1}^{n} a_kx_kr_k,$$

where

$$r_k = \frac{1}{k\sqrt[k]{a_1a_2...a_k}} + \frac{1}{(k+1)\sqrt[k+1]{a_1a_2...a_{k+1}}} + ... + \frac{1}{n\sqrt[n]{a_1a_2...a_n}}.$$

Finally, we will determine numbers $(a_1, a_2, ..., a_n)$ for which $a_kr_k \le e\ \forall k \in \{1, 2, ..., n\}$. The form of r_k suggests to find a_k for which $\sqrt[k]{a_1a_2...a_k}$ can be simplified. Intuitively, we choose $a_1 = 1$ and $a_k = \dfrac{k^k}{(k-1)^{k-1}}$. With these values we have $\sqrt[k]{a_1a_2...a_k} = k\ \forall k \in \{1, 2, ..., n\}$. Therefore, for all $k > 1$, we have

$$\begin{aligned} a_kr_k &= \frac{k^k}{(k-1)^{k-1}}\left(\frac{1}{k^2} + \frac{1}{(k+1)^2} + ... + \frac{1}{n^2}\right) \\ &\le \frac{k^k}{(k-1)^{k-1}}\left(\frac{1}{k-1} - \frac{1}{k} + \frac{1}{k} - \frac{1}{k+1} + ... + \frac{1}{n-1} - \frac{1}{n}\right) \\ &= \left(1 + \frac{1}{k-1}\right)^{k-1} \le e. \end{aligned}$$

For $k = 1$, we also have

$$a_1r_1 = \frac{1}{1^2} + \frac{1}{2^2} + ... + \frac{1}{n^2} \le 2 \le e.$$

∇

Chapter 6

The Method of Balanced Coefficients

In many problems, grouping terms in order to use classical inequalities is not easy, especially for non-symmetric inequalities. In these cases, the coefficients of similar terms are normally not equal to each other and therefore we not only have to use basic inequalities properly but also have to case take care of the equality so that this case is maintained throughout the solution. How should use deal with this matter? Generally, we need to use additional variables then solve the equations to find out the original variables in the end. This method is called *the method of balanced coefficients.*

In order to see this important method at work, let's start with the following simple example

Example 6.0.8. *Let x, y, z be positive real numbers and $xy+yz+zx=1$. Prove that*

$$10x^2+10y^2+z^2 \geq 4.$$

SOLUTION. Let's first see a nice, short and a bit magic solution, before we give a general and natural solution. By **AM-GM** inequality, we deduce that $2x^2+2y^2 \geq 4xy$, $8x^2+1/2z^2 \geq 4xz$ and $8y^2+1/2z^2 \geq 4yz$. Adding up these inequalities, we have

$$10x^2+10y^2+z^2 \geq 4(xy+yz+zx)=4.$$

Equality holds for

$$\begin{cases} x = y \\ 4x = z \\ 4y = z \end{cases} \quad \Leftrightarrow \quad \begin{cases} x = y = 1/3 \\ z = 4/3. \end{cases}$$

∇

This solution really needs to be examined. Some questions are posed: why could we separate $10 = 2 + 8$? Is it lucky and accidental? Is it obvious? If we separate 10 in other ways, such that $10 = 3 + 7$ or $10 = 4 + 6$, do we go get the same final result? In fact, every other separation is not effective and the separation $10 = 2 + 8$ is not lucky. Not surprisingly, we have already used the method of balanced coefficients, which is hidden in this apparently *obvious* solution. Let's continue with two major means of applying this method: balancing coefficients by **AM-GM** inequality and balancing coefficients by **Cauchy-Schwarz** inequality.

6.1 Balancing coefficients by AM-GM inequality

No matter how familiar you are with balancing coefficients, non-symmetric inequalities with a chaotic of coefficients always cause a lot of difficulties. Therefore, using this method deftly can help you avoid a lot o computations. The following general proof will explain how we got the solution in example 6.0.8.

Example 6.1.1. *Let k be a positive real number. Find the minimum of*

$$k(x^2 + y^2) + z^2,$$

where x, y, z are three positive real numbers such that $xy + yz + zx = 1$.

SOLUTION. We separate $k = l + (k - l)$ (with the condition $0 \le l \le k$) and apply **AM-GM** inequality in the following form

$$lx^2 + ly^2 \ge 2lxy$$
$$(k - l)x^2 + 1/2z^2 \ge \sqrt{2(k - l)}xz$$
$$(k - l)y^2 + 1/2z^2 \ge \sqrt{2(k - l)}yz.$$

These results, combined, yield that

$$k(x^2 + y^2) + z^2 \ge 2lxy + \sqrt{2(k - l)}(xz + yz).$$

6.2 Balancing coefficients by Cauchy-Schwarz and Hölder inequalities

A great difference between **Cauchy-Schwarz** inequality, **Hölder** inequality and **AM-GM** inequality is the case when equality holds. This feature also leads to different ways to balance coefficients in these inequalities. Let's contemplate the following examples to get an overview.

Example 6.2.1. *Suppose that x, y, z are three positive real numbers verifying $x+y+z=3$. Find the minimum of the expression*

$$x^4 + 2y^4 + 3z^4.$$

SOLUTION. Let a, b, c be three positive real numbers such that $a+b+c=3$. According to **Hölder** inequality, we obtain

$$(x^4 + 2y^4 + 3z^4)(a^4 + 2b^4 + 3c^4)^3 \geq (a^3x + 2b^3y + 3c^3z)^4 \quad (\star)$$

We will choose a, b, c such that $a^3 = 2b^3 = 3c^3 = k^3$, and if then

$$x^4 + 2y^4 + 3z^4 \geq \frac{k^{12}(x+y+z)^4}{(a^4+2b^4+3c^4)^3} = \frac{(3k^3)^4}{(a^4+2b^4+3c^4)^3} \quad (\star\star)$$

The equality in $(\star)$ happens when $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$. Because $x+y+z = a+b+c = 3$, we get $a = x, b = y, c = z$. We find that $k = \frac{3}{1+\sqrt[3]{2}+\sqrt[3]{3}}$ and $a = k, b = \sqrt[3]{2}k, c = \sqrt[3]{3}k$. The minimum of $x^4 + 2y^4 + 3z^4$ is given by $(\star\star)$.

Comment. The following general problem can be solved by the same method

★ *Suppose that $x_1, x_2, ..., x_n$ are positive real numbers with sum n and $a_1, a_2, ..., a_n$ are positive real constants. For each positive integer m, find the minimum of*

$$a_1x_1^m + a_2x_2^m + ... + a_nx_n^m.$$

By **Hölder** inequality, we find that the minimum of this expression is na^{m-1} where

$$a = \frac{n}{\frac{1}{\sqrt[m-1]{a_1}} + \frac{1}{\sqrt[m-1]{a_2}} + ... + \frac{1}{\sqrt[m-1]{a_1}}}.$$

∇

Example 6.2.2. *Let $a_1, a_2, ..., a_n$ be positive real numbers. Prove that*

$$\frac{1}{a_1} + \frac{2}{a_1+a_2} + ... + \frac{n}{a_1+a_2+...+a_n} < 2\left(\frac{1}{a_1} + \frac{1}{a_2} + ... + \frac{1}{a_n}\right).$$

SOLUTION. Let $x_1, x_2, ..., x_n$ be positive real numbers (we will determine them at the end). According to **Cauchy-Schwarz** inequality, we have

$$(a_1 + a_2 + ... + a_k)\left(\frac{x_1^2}{a_1} + \frac{x_2^2}{a_2} + ... + \frac{x_k^2}{a_k}\right) \geq (x_1 + x_2 + ... + x_k)^2$$

$$\Rightarrow \frac{k}{a_1 + a_2 + ... + a_k} \leq \frac{k}{(x_1 + x_2 + ... + x_k)^2}\left(\frac{x_1^2}{a_1} + \frac{x_2^2}{a_2} + ... + \frac{x_n^2}{a_n}\right).$$

Constructing similar results for all $k \in \{1, 2, ..., n\}$ then adding up all of them, we get

$$\frac{1}{a_1} + \frac{2}{a_1 + a_2} + ... + \frac{n}{a_1 + a_2 + ... + a_n} \leq \frac{c_1}{a_1} + \frac{c_2}{a_2} + ... + \frac{c_n}{a_n},$$

where c_k, $k \in \{1, 2, ..., n\}$, is determined by

$$c_k = \frac{kx_k^2}{(x_1 + x_2 + ... + x_k)^2} + \frac{(k+1)x_k^2}{(x_1 + x_2 + ... + x_{k+1})^2} + ... + \frac{nx_k^2}{(x_1 + x_2 + ... + x_n)^2}.$$

We have to find x_k for which $c_k \leq 2\ \forall 1 \leq k \leq n$. We simply choose $x_k = k$ then

$$\begin{aligned} c_k &= k^2\left(\sum_{j=k}^{n} \frac{j}{(1 + 2 + ... + j)^2}\right) = 4k^2\left(\sum_{j=k}^{n} \frac{1}{j(j+1)^2}\right) \leq 2k^2\left(\sum_{j=k}^{n} \frac{2j+1}{j^2(j+1)^2}\right) \\ &= 2k^2\left(\sum_{j=k}^{n} \frac{1}{j^2} - \sum_{j=k}^{n} \frac{1}{(j+1)^2}\right) = 2k^2\left(\frac{1}{k^2} - \frac{1}{(n+1)^2}\right) < 2. \end{aligned}$$

The proof is completed.

∇

Example 6.2.3. *Let $x_1, x_2, ..., x_n$ be positive real numbers. Prove that*

$$x_1^2 + \left(\frac{x_1 + x_2}{2}\right)^2 + ... + \left(\frac{x_1 + x_2 + ... + x_n}{n}\right)^2 \leq 4\left(x_1^2 + x_2^2 + ... + x_n^2\right).$$

SOLUTION. Let $\alpha_1, \alpha_2, ..., \alpha_n$ be positive real numbers. According to **Cauchy-Schwarz** inequality, we have

$$\left(\frac{x_1^2}{\alpha_1} + \frac{x_2^2}{\alpha_2} + ... + \frac{x_k^2}{\alpha_k}\right)(\alpha_1 + \alpha_2 + ... + \alpha_k) \geq (x_1 + x_2 + ... + x_n)^2.$$

Rewrite this inequality to

$$\begin{aligned} \left(\frac{x_1 + x_2 + ... + x_k}{k}\right)^2 \leq \frac{\alpha_1 + \alpha_2 + ... + \alpha_k}{k^2\alpha_1} \cdot x_1^2 + \frac{\alpha_1 + \alpha_2 + ... + \alpha_k}{k^2\alpha_2} \cdot x_2^2 + ... \\ + \frac{\alpha_1 + \alpha_2 + ... + \alpha_k}{k^2\alpha_k} \cdot x_k^2. \end{aligned}$$

Constructing similar results for all $k \in \{1,2,...,n\}$ then adding up all of them, we get

$$x_1^2+\left(\frac{x_1+x_2}{2}\right)^2+...+\left(\frac{x_1+x_2+...+x_n}{n}\right)^2 \le \gamma_1 x_1^2+\gamma_2 x_2^2+...+\gamma_n x_n^2$$

in which each coefficient γ_k is defined by

$$\gamma_k=\frac{\alpha_1+\alpha_2+...+\alpha_k}{k^2\alpha_k}+\frac{\alpha_1+\alpha_2+...+\alpha_{k+1}}{(k+1)^2\alpha_k}+...+\frac{\alpha_1+\alpha_2+...+\alpha_n}{n^2\alpha_k}.$$

The solution is completed if there is a sequence $(\alpha_1,\alpha_2,...,\alpha_n)$ such that $\gamma_k \le 4\ \forall k \in \{1,2,...,n\}$. We choose $\alpha_k=\sqrt{k}-\sqrt{k-1}$, then $\alpha_1+\alpha_2+...+\alpha_k=\sqrt{k}$. In this case

$$\gamma_k=\frac{1}{\alpha_k}\left(\frac{1}{k^{3/2}}+\frac{1}{(k+1)^{3/2}}+...+\frac{1}{n^{3/2}}\right).$$

Notice that $\sqrt{(k-\frac{1}{2})(k+\frac{1}{2})}\left(\sqrt{k-\frac{1}{2}}+\sqrt{k+\frac{1}{2}}\right) \le 2k^{3/2}$, so

$$\frac{1}{k^{3/2}} \le \frac{\sqrt{k+\frac{1}{2}}-\sqrt{k-\frac{1}{2}}}{\sqrt{(k-\frac{1}{2})(k+\frac{1}{2})}}=\frac{1}{\sqrt{k-\frac{1}{2}}}-\frac{1}{\sqrt{k+\frac{1}{2}}}.$$

We conclude that

$$\gamma_k=\frac{1}{\alpha_k}\left(\sum_{j=k}^{n}\frac{1}{j^{3/2}}\right) \le \frac{1}{\alpha_k}\left(\sum_{j=k}^{n}\frac{1}{\sqrt{j-\frac{1}{2}}}-\sum_{j=k}^{n}\frac{1}{\sqrt{j+\frac{1}{2}}}\right)$$
$$\le \frac{2}{\alpha_k\sqrt{k-\frac{1}{2}}}=\frac{2(\sqrt{k}+\sqrt{k-1})}{\sqrt{k-\frac{1}{2}}}<4.$$

Comment. The following similar result is left as an exercise.

★ *Let $x_1,x_2,...,x_n$ be positive real numbers. Prove that*

$$x_1^3+\left(\frac{x_1+x_2}{2}\right)^3+...+\left(\frac{x_1+x_2+...+x_n}{n}\right)^3 \le \frac{27}{8}\left(x_1^3+x_2^3+...+x_3^3\right).$$

∇

Example 6.2.4. *Find the best value of $t=t(n)$ (smallest) for which the following inequality is true for all real numbers $x_1,x_2,...,x_n$*

$$x_1^2+(x_1+x_2)^2+...+(x_1+x_2+...+x_n)^2 \le t(x_1^2+x_2^2+...+x_n^2).$$

(MYM 2004)

SOLUTION. Let $c_1, c_2, ..., c_n$ be positive real numbers (which will be chosen later). According to **Cauchy-Schwarz** inequality, we have

$$\left(\sum_{i=1}^{k} x_i\right)^2 \le S_k \left(\sum_{i=1}^{k} \frac{x_i^2}{c_i}\right) \qquad (\bigstar)$$

where $S_1, S_2, ..., S_n$ are determined as

$$S_k = \sum_{i=1}^{k} c_i,\ k \in \{1, 2, ..., n\}.$$

According to $(\star)$, we infer that (after adding up similar results for all $k \in \{1, 2, ..., n\}$)

$$\sum_{k=1}^{n}\left(\sum_{i=1}^{k} x_i\right)^2 \le \sum_{k=1}^{n}\left(\sum_{j=k}^{n} \frac{S_j}{c_j}\right) x_i^2.$$

We will choose coefficients $c_1, c_2, ..., c_n$ such that

$$\frac{S_1 + S_2 + ... + S_n}{c_1} = \frac{S_2 + S_3 + ... + S_n}{c_2} = \cdots = \frac{S_n}{c_n} = t.$$

After some computations, we find

$$c_i = \sin i\alpha - \sin(i-1)\alpha\ \forall k \in \{1, 2, ..., n\},$$

where $\alpha = \dfrac{\pi}{2n+1}$. So $t = \dfrac{1}{4\sin^2 \frac{\pi}{2(2n+1)}}$ and we conclude that

$$\sum_{k=1}^{n}\left(\sum_{i=1}^{k} x_i\right)^2 \le t\left(\sum_{k=1}^{n} x_i^2\right) = \frac{1}{4\sin^2 \frac{\pi}{2(2n+1)}}\left(\sum_{k=1}^{n} x_i^2\right).$$

Chapter 7

Derivative and Applications

We will discuss now one of the most important concepts of Mathematics. Just by realizing the great impact of derivatives on the development of Mathematics, you will understand how widely and deeply derivatives are affecting the world of inequalities nowadays. Therefore it's necessary for you to comprehend this concept and master it as one expert.

7.1 Derivative of one-variable functions

The principal objective of derivatives is to help examine one-variable functions. To find maximum or minimum of a function which has only one variable, derivatives seems to be infallible work. That's the reason we believe that every inequality in one variable is either solved by derivatives or impossible to be solved.

The application of derivatives to one-variable functions is not restricted one-variable inequalities. In fact, can help you to solve many $n-$variable inequalities as you will see in the following pages.

Example 7.1.1. *Find the minimum value of x^x, if x is a positive real number.*

SOLUTION. Consider the function $f(x) = x^x = e^{x\ln x}$. Its derivative is $f'(x) = e^{x\ln x}(\ln x + 1)$. Clearly, $f'(x) = 0 \Leftrightarrow \ln x = -1 \Leftrightarrow x = 1/e$. In $\left(0, \frac{1}{e}\right]$, $f(x)$ is strictly decreasing and in $\left[\frac{1}{e}, +\infty\right)$, $f(x)$ is strictly increasing. That means

$$\min_{x\in\mathbb{R}} f(x) = f\left(\frac{1}{e}\right) = \frac{1}{e^{\frac{1}{e}}}.$$

∇

Example 7.1.2. *Let a, b, c be positive real numbers. Prove that*

$$\frac{a^3}{b^3+c^3}+\frac{b^3}{c^3+a^3}+\frac{c^3}{a^3+b^3} \geq \frac{a^2}{b^2+c^2}+\frac{b^2}{c^2+a^2}+\frac{c^2}{a^2+b^2}.$$

SOLUTION. We will solve the general problem for all real numbers $s \geq t \geq 0$

$$\frac{a^s}{b^s+c^s}+\frac{b^s}{c^s+a^s}+\frac{c^s}{a^s+b^s} \geq \frac{a^t}{b^t+c^t}+\frac{b^t}{c^t+a^t}+\frac{c^t}{a^t+b^t}.$$

It's enough to prove that the following function is increasing by derivative

$$f(x)=\frac{a^x}{b^x+c^x}+\frac{b^x}{c^x+a^x}+\frac{c^x}{a^x+b^x}.$$

Indeed, after a bit of calculation

$$\begin{aligned} f'(x) &= \sum_{cyc} \frac{a^x \cdot \ln a \cdot (b^x+c^x) - a^x(b^x \cdot \ln b - c^x \cdot \ln c)}{(b^x+c^x)^2} \\ &= \sum_{cyc} \frac{a^x b^x (a^x-b^x)(\ln a - \ln b)(2c^x+a^x+b^x)}{(a^x+b^2)^2(b^x+c^x)^2} \geq 0. \end{aligned}$$

Comment. The following general problem can be solved in a similar fashion

★ *Let $a_1, a_2, ..., a_n$ be positive real numbers with sum 1. Prove that for all real numbers $s \geq t \geq 0$, we have*

$$\left(\frac{a_1}{1-a_1}\right)^s+\left(\frac{a_2}{1-a_2}\right)^s+...+\left(\frac{a_n}{1-a_n}\right)^s \geq \left(\frac{a_1}{1-a_1}\right)^t+\left(\frac{a_2}{1-a_2}\right)^t+...+\left(\frac{a_n}{1-a_n}\right)^t.$$

∇

Example 7.1.3. *Let a, b, c, d be positive real numbers. Prove that*

$$\sqrt{\frac{ab+ac+ad+bc+bd+cd}{6}} \geq \sqrt[3]{\frac{abc+bcd+cda+dab}{4}}.$$

SOLUTION. Consider the function

$$f(x)=(x-a)(x-b)(x-c)(x-d)=x^4-Ax^3+Bx^2-Cx+D$$

where

$$A=\sum_{sym} a, \quad B=\sum_{sym} ab, \quad C=\sum_{sym} abc, \quad D=abcd.$$

Since the equation $f(x)=0$ has 4 positive real roots, we infer that (by **Rolle** theorem) the equation $f'(x)=0$ has 3 positive real roots, too. Denote these roots m, n, p $(m, n, p>0)$, then

$$f'(x)=4(x-m)(x-n)(x-p)=4x^3-4(m+n+p)x^2+4(mn+np+pm)x-4mnp.$$

Notice that we also have $f'(x) = 4x^3 - 3Ax^2 + 2Bx - C$, so $B = 2(mn + np + pm)$ and $C = 4mnp$. By **AM-GM** inequality, we conclude that

$$\sqrt{\frac{B}{6}} = \sqrt{\frac{mn + np + pm}{3}} \geq \sqrt[3]{mnp} = \sqrt[3]{\frac{C}{4}}.$$

Comment. Suppose that $x_1, x_2, ..., x_n$ are positive real numbers and $d_1, d_2, ..., d_n$ are the polynomials defined as

$$d_k = \frac{1}{C_n^k} \sum_{sym} x_1 x_2 ... x_n.$$

With the same method, we can prove the following results

★ (**Newton** inequality). *For all positive real numbers* $x_1, x_2, ..., x_n$

$$d_{k+1} d_{k-1} \leq d_k^2.$$

★ (**Maclaurin** inequality). *For all positive real numbers* $x_1, x_2, ..., x_n$

$$d_1 \geq \sqrt{d_2} \geq ... \geq \sqrt[k]{d_k} \geq ... \geq \sqrt[n]{d_n}.$$

∇

Example 7.1.4. *Prove that* $0 \leq a \leq 1 \leq b \leq 3 \leq c \leq 4$ *if* a, b, c *are real numbers satisfying the conditions*

$$a \leq b \leq c, a + b + c = 6, ab + bc + ca = 9.$$

(British MO)

Solution. Denote $p = abc$ and consider the function

$$f(x) = (x - a)(x - b)(x - c) = x^3 - 6x^2 + 9x - p.$$

We have $f'(x) = 3x^2 - 12x + 9 = 3(x-1)(x-3)$. Therefore $f'(x) = 0$ or $x = 1 \vee x = 3$. Because $f(x)$ has three roots $a \leq b \leq c$, we infer

$$1 \leq b \leq 3 \ , \ f(1)f(3) \leq 0.$$

Note that $f(1) = f(4) = 4 - p$ and $f(0) = f(3) = -p$, so we have $0 \leq p \leq 4$. That implies $f(1) = f(4) \geq 0$ and $f(0) = f(3) \leq 0$. If $f(0) = f(3) = 0$ then $a = 0, b = c = 3$ and the desired result is obvious. If $f(1) = f(4) = 0$ then $a = b = 1, c = 4$ and the desired result is obvious as well. Otherwise, we must have $f(0)f(1) < 0$, $f(1)f(3)<0$, $f(3)f(4)<0$ and therefore $a \in (0, 1)$, $b \in (1, 3)$, $c \in (3, 4)$. The proof is finished.

∇

7.2 Derivative of n-Variable Functions

If you feel that one variable functions are easy already, if you feel their extremums can always be easily found by derivative, let's take a glimpse at functions of more variables. Although it's much more difficult to find the extremums of functions of n variables, we approach these problems is the same as what we do with one-variable functions. If there are some conditions that restrict variables, try to change and eliminate these conditions and make a new expression where every variable is independent from each other, then find the extremum of this new expression as a one-variable function of each variable. Let's see the following examples to clarify the method.

Example 7.2.1. *Let a, b, c be positive real numbers. Prove that*

$$a^3 + b^3 + c^3 + 3abc \geq ab(a+b) + bc(b+c) + ca(c+a).$$

SOLUTION. WLOG, assume that $a \geq b \geq c$. Consider the function of a: $f(a) = a^3 + b^3 + c^3 + 3abc - ab(a+b) - bc(b+c) - ca(c+a)$. We have

$$f'(a) = 3a^2 + 3bc - 2ab - b^2 - 2ac - c^2.$$

Notice that $f''(a) = 6a - 2b - 2c \geq 0$ and $f''(b) \geq 0$, so $f'(a) \geq f'(b) = c(b-c) \geq 0$. Also, $f'(x) \geq f'(b) \geq 0$, $(\forall) x \in (b, a)$, since f'' is linear, therefore positive on (b, a). That implies $f(a) \geq f(b) = c(b-c)^2 \geq 0$. The proof is finished.

∇

Example 7.2.2. *Let a, b, c, d be positive real numbers such that*

$$2(ab + bc + cd + da + ac + bd) + abc + bcd + cda + dab = 16.$$

Prove the following inequality

$$a + b + c + d \geq \frac{2}{3}(ab + bc + cd + da + ac + bd).$$

(Vietnam MO 1996)

SOLUTION. By a similar reasoning as in example 7.1.3, we deduce that there exist positive real numbers x, y, z for which

$$\sum_{sym} a = \frac{4}{3} \sum_{sym} x \, , \quad \sum_{sym} ab = 2 \sum_{sym} xy \, , \quad \sum_{sym} abc = 4xyz.$$

It remains to prove that if $xy + yz + zx + xyz = 4$ then $x + y + z \geq xy + yz + zx$.

Certainly, there exist two numbers, say x and y, both greater than 1 or both smaller than 1. In this case, $(x-1)(y-1) \geq 0 \Rightarrow xy+1 \geq x+y$. We denote $m = x+y, n = xy$ then $z = \dfrac{4-n}{m+n}$. If $m \geq 4$ then $x+y+z \geq 4 \geq xy+zx+zx$ and the conclusion follows. Otherwise, $m-1 \leq n \leq \dfrac{m^2}{4} \leq 4$ and we need to prove

$$m + \frac{4-n}{m+n} \geq \frac{(4-n)m}{m+n} + n$$

$$\Leftrightarrow \; f(n) = -n^2 + n(m-1) + m^2 - 4m + 4 \geq 0.$$

Notice that $f'(n) = -2n + m - 1 \leq -n + m - 1 \leq 0$, so $f(n)$ is decreasing, therefore

$$f(n) \geq f\left(\frac{m^2}{4}\right) = \frac{(16-m^2)(m-2)^2}{16} \geq 0.$$

We are done. Equality holds for $a = b = c = d = 1$.

∇

Example 7.2.3. *Let a, b, c be non-negative real numbers. Prove that*

$$a^3 + b^3 + c^3 + 9abc + 4(a+b+c) \geq 8(ab+bc+ca).$$

(Le Trung Kien)

SOLUTION. We denote

$$f(b) = b^3 + b(4 + 9ac - 8a - 8c) + a^3 + c^3 + 4(a+c) - 8ac.$$

By **AM-GM** inequality, $(a^3+4a) + (c^3+4c) \geq 4a^2 + 4c^2 \geq 8ac$, so the problem is proved in case $4 + 9ac \geq 8(a+c)$, with equality for $a = c = 2, b = 0$ and $a = b = c = 0$. Otherwise, let $x = a + c, y = ac$ then, $8x \geq 9y + 4$. Notice that

$$f'(b) = 3b^2 - (8x - 9y - 4), \text{ so } f'(b) = 0 \Leftrightarrow b = \sqrt{\frac{8x-9y-4}{3}}$$

and therefore

$$f(b) \geq f\left(\sqrt{\frac{8x-9y-4}{3}}\right) = \frac{-2}{3\sqrt{3}}(8x-9y-4)^{3/2} + x^3 + 4x - 3xy - 8y = g(y).$$

Because $y \leq \dfrac{x^2}{4}$ and $y \leq \dfrac{8x-4}{9}$ (that also means $x \geq \dfrac{1}{2}$), as get $y \leq \min\left(\dfrac{x^2}{4}, \dfrac{8x-4}{9}\right) = t$.

$$g'(y) = 3\sqrt{3(8x-9y-4)} - (3x+8) \leq 3\sqrt{3(8x-4)} - (3x+8) < 0.$$

Thus $g(y)$ is strictly decreasing and therefore $g(y) \geq g(t)$. If $t = \frac{8x-4}{9}$ then $g(t) = x^3 + 4x - 3xt - 8t \geq 0$ (or equivalently $a^3 + c^3 + 4(a+c) - 8ac \geq 0$). It's enough to consider the remaining case $t = \frac{x^2}{4}$ and prove that $g(t) \geq 0$. Denote $s = \frac{x}{2}$, the inequality $g\left(\frac{x^2}{4}\right) = g(s^2) \geq 0$ is equivalent to

$$h(s) = 2s^3 - 8s^2 + 8s - \frac{2}{3\sqrt{3}}(16s - 9s^2 - 4)^{3/2} \geq 0.$$

Because $h'(s) = 6s^2 - 16s + 8 - (16 - 18s)\sqrt{\frac{16s - 9s^2 - 4}{3}}$, if $h'(s) = 0$ for some s then we must have $\frac{8}{9} \leq s \leq \frac{8+\sqrt{28}}{9}$ and

$$3(2s^2 - 8s + 4)^2 = (9 - 8s)^2(16s - 9s^2 - 4)$$

$$\Leftrightarrow \quad (s-1)(189s^3 - 495s^2 + 372s - 76) = 0.$$

Notice that the equation $189s^3 - 495s^2 + 372s - 76 = 0$ has exactly one real root in the interval $\left[\frac{8}{9}, \frac{8+\sqrt{28}}{9}\right]$, so it's easy to infer that $h(s) \geq h(1) = 0$. Equality holds for $a = b = c = 1$ or $a = b = c = 0$ or $a = b = 2, c = 0$ up to permutation.

∇

Example 7.2.4. *Suppose n is an integer greater than 2 and that the n positive real numbers $x_1, x_2, ..., x_n$ satisfy the condition*

$$(x_1 + x_2 + ... + x_n)\left(\frac{1}{x_1} + \frac{1}{x_2} + ... + \frac{1}{x_n}\right) \leq (n + \sqrt{10} - 3)^2.$$

Prove that every 3-uple (x_i, x_j, x_k) $(1 \leq i < j < k \leq n$ and $i, j, k \in \mathbb{N})$ can be the length-sides of a triangle.

(Improved IMO 2004, B1)

SOLUTION. It suffices to prove the following result (that directly solves this problem)

Suppose that $x_1 \geq x_2 \geq ... \geq x_n > 0$ are real numbers verifying $x_1 > x_2 + x_3$, then

$$(x_1 + x_2 + ... + x_n)\left(\frac{1}{x_1} + \frac{1}{x_2} + ... + \frac{1}{x_n}\right) > (n + \sqrt{10} - 3)^2.$$

Indeed, we will prove it by induction. For $n = 3$, the inequality becomes

$$(x_1 + x_2 + x_3)\left(\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}\right) > 10 \Leftrightarrow \sum_{cyc} x_1x_2(x_1 + x_2) > 7x_1x_2x_3.$$

Let $f(x_1) = \sum_{cyc} x_1x_2(x_1 + x_2) - 7x_1x_2x_3$ then

$$f'(x_1) = 2x_1(x_2 + x_3) + x_2^2 + x_3^2 - 7x_2x_3 > 2(x_2 + x_3)^2 + x_2^2 + x_3^2 - 7x_2x_3 > 0.$$

This implies that $f(x_1) \geq f(x_2 + x_3) = (x_2 + x_3)^2(x_2 - x_3) > 0$. We are done.

Now return to the problem of $n + 1$ variables (with the supposition that it is true for n variables). We need to prove that

$$(x_1 + x_2 + ... + x_{n+1}) \left(\frac{1}{x_1} + \frac{1}{x_2} + ... + \frac{1}{x_{n+1}}\right) > (n + \sqrt{10} - 2)^2.$$

Denote $A = \sum_{i=1}^{n} x_i$ and $B = \sum_{i=1}^{n} x_i^{-1}$, then $AB > (n + \sqrt{10} - 3)^2$ by hypothesis. Let $x = x_{n+1}$ then $A/B > x_n^2 \geq x^2 \Rightarrow \sqrt{A/B} > x$. Denote

$$f(x) = (x + A)\left(\frac{1}{x} + B\right) = Bx + \frac{A}{x} + 1 + AB.$$

We have of course $f'(x) = B - \dfrac{A}{x^2}$ and therefore $f'(x) = 0 \Leftrightarrow x = \sqrt{A/B}$. Thus

$$f(x) \geq f\left(\sqrt{\frac{A}{B}}\right) = \left(1 + \sqrt{AB}\right)^2 > (n + \sqrt{10} - 2)^2.$$

The inductive step is completed and we are done.

∇

Example 7.2.5. *Let a, b, c be positive real numbers such that $12 \geq 21ab + 2bc + 8ca$. Prove that*

$$\frac{1}{a} + \frac{2}{b} + \frac{3}{c} \leq \frac{15}{2}.$$

(Tran Nam Dung, Vietnam TST 2001)

SOLUTION. Although this problem has been solved in the previous part by balancing coefficients, it's seem to be more intuitive to give a solution by derivatives. Let $x = \dfrac{1}{a}, y = \dfrac{2}{b}, z = \dfrac{3}{c}$. We will prove an equivalent problem as follows.

If $x, y, z > 0$ and $12xyz \geq 2x + 8y + 21z$ then $P(x, y, z) = x + 2y + 3z \leq \dfrac{15}{2}$.

Indeed, by hypothesis we have $z(12xy-21) \geq 2x + 8y > 0$. So we infer that $12xy \geq 21$, or $x > \dfrac{7}{4y}$ and $z \geq \dfrac{2x + 8y}{12xy - 21}$. That means

$$P(x, y, z) \geq x + 2y + \frac{2x + 8y}{4xy - 7} = f(x).$$

We have of course

$$f'(x) = \frac{16x^2y^2 - 56xy - 32y^2 + 35}{(4xy-7)^2}.$$

In the range $\left(\frac{7}{4y}, +\infty\right)$, the equation $f'(x) = 0$ has only one root $x = x_0 = \frac{7}{4y} + \frac{\sqrt{32y^2+14}}{4y}$. At $x = x_0$, $f'(x)$ changes its sign from negative to positive, so $f(x)$ attains the minimum at x_0. It implies

$$f(x) \ge f(x_0) = 2x_0 - \frac{5}{4y} \Rightarrow P(x,y,z) \ge f(x) + 2y \ge f(x_0) + 2y = g(y)$$

where

$$g(y) = 2y + \frac{9}{4y} + \frac{1}{2y}\sqrt{32y^2+14}.$$

A simple calculation will show that

$$g'(y) = 0 \Leftrightarrow (8y^2 - 9)\sqrt{32y^2+14} - 28 = 0.$$

Denote $t = \sqrt{32y^2+14}$, then $t > 0$. The above equation becomes $t^3 - 50t - 112 = 0$. This equation has only one positive real root $t = 8$, or $y = y_0 = \frac{5}{4}$, thus $g'\left(\frac{5}{4}\right) = 0$. In the range $y > 0$, at $y = y_0$, $g'(y)$ changes its sign from negative to positive, hence $g(y)$ attains the minimum at y_0. So we conclude that $g(y_0) = g\left(\frac{5}{4}\right) = \frac{15}{2}$ and then $P(x,y,z) \ge g(y) \ge g(y_0) = \frac{15}{2}$. Equality holds for $y = \frac{5}{4}, x = 3, z = \frac{2}{3}$ or $a = \frac{1}{3}, b = \frac{4}{5}, c = \frac{3}{2}$. We are done.

∇

Example 7.2.6. *Let a, b, c be three positive real numbers such that*

$$(a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) = 16.$$

Find the minimum and maximum value of

$$P = \frac{a}{b} + \frac{b}{c} + \frac{c}{a}.$$

(Pham Kim Hung)

SOLUTION. We will first solve the problem of finding the minimum value. To find this value we can assume that $a \ge b \ge c$ because in this case, we have

$$\sum_{cyc} \frac{a}{b} - \sum_{cyc} \frac{b}{a} = \frac{(a-b)(a-c)(c-b)}{abc} \le 0.$$

Denote $x = \frac{a}{b} \geq 1, y = \frac{b}{c} \geq 1$. The hypothesis yields that

$$x + y + \frac{1}{xy} + \frac{1}{x} + \frac{1}{y} + xy = 13.$$

Let now $x + y = s$ and $xy = t$, then $P = s + \frac{1}{t}$ and

$$s + t + \frac{s}{t} + \frac{1}{t} = 13 \Rightarrow s = \frac{13t - t^2 - 1}{t + 1}.$$

This relation implies that

$$P = f(t) = \frac{13t - t^2 - 1}{t + 1} + \frac{1}{t}.$$

Then

$$\begin{aligned} f'(t) &= \frac{(13 - 2t)(t + 1) - (13t - t^2 - 1)}{(t + 1)^2} - \frac{1}{t^2} \\ &= \frac{-t^2 - 2t + 14}{(t + 1)^2} - \frac{1}{t^2} = \frac{15}{(t + 1)^2} - \frac{t^2 + 1}{t^2}. \end{aligned}$$

It's easy to infer that

$$f'(t) = 0 \Leftrightarrow (t^2 + 1)(t + 1)^2 = 15t^2 \Leftrightarrow \left(t + \frac{1}{t} + 1\right)^2 = 16 \Leftrightarrow t + \frac{1}{t} = 3.$$

Because $t = ab \geq 1$, $f'(t) = 0 \Leftrightarrow t = t_0 = \frac{3 + \sqrt{5}}{2}$. Moreover, the supposition $x, y \geq 1$ shows that $t + 1 - s = (x - 1)(y - 1) \geq 0$ or $t + 1 \geq s$. It implies

$$\frac{13t - t^2 - 1}{t + 1} \leq t + 1 \Leftrightarrow 2t^2 - 11t + 2 \geq 0 \Rightarrow t \geq \frac{11 + \sqrt{105}}{4} > t_0.$$

Thus $f(t)$ is strictly decreasing. To find the minimum of $f(t)$, it's enough to find the maximum of t. Notice that $s^2 = (x + y)^2 \geq 4t$, so we obtain

$$(13t - t^2 - 1)^2 \geq 4t(t + 1)^2 \Rightarrow \left(13 - t - \frac{1}{t}\right)^2 \geq 4\left(t + \frac{1}{t} + 2\right)$$

$$\Rightarrow \left(t + \frac{1}{t}\right)^2 - 30\left(t + \frac{1}{t}\right) + 161 \geq 0 \Leftrightarrow \left(t + \frac{1}{t} - 7\right)\left(t + \frac{1}{t} - 23\right) \geq 0.$$

Furthermore, $t + \frac{1}{t} < 23$ (because $1 \leq t \leq 13$), so we must have $t + \frac{1}{t} \geq 7$ or

$$t^2 - 7t + 1 \leq 0 \Leftrightarrow t \leq \frac{7 + 3\sqrt{5}}{2}.$$

For this value of t $\left(t = \dfrac{7+\sqrt{5}}{2}\right)$, we have $s = 2\sqrt{t} = \sqrt{14+6\sqrt{5}} = 3+\sqrt{5}$ and conclude that

$$\min f(t) = s + \frac{1}{t} = 3 + \sqrt{5} + \frac{2}{7+3\sqrt{5}} = \frac{13-\sqrt{5}}{2},$$

with equality for $\dfrac{a}{b} = \dfrac{b}{c} = \sqrt{t} = \dfrac{3+\sqrt{5}}{2}$ and permutations.

Similarly, with the same method, we find that $\max f(t) = \dfrac{13+\sqrt{5}}{2}$, with equality for $\dfrac{a}{b} = \dfrac{b}{c} = \dfrac{3-\sqrt{5}}{2}$ and permutations.

∇

Each problem has its own features. If you can fathom the particular features, you can find particular methods to solve them. For example, you are aware of using balancing coefficients to solve non-cyclic inequalities or using symmetric separation to solve symmetric inequalities. These solutions are, generally, technical and hard to figure out. However, derivatives are different. Although solutions by derivatives are, in fact, coarse and require some long computations, they are very natural. That's the reason why derivatives are so important and indispensable in every part of the field of inequalities and mathematics as well.

Chapter 8

A note on symmetric inequalities

In the beautiful world of inequalities, symmetric inequalities seem to be the most important and most beloved kind. This kind of inequalities also plays a major part in Mathematics Contests all over the world and the overview in this chapter is really necessary. Although there are a lot of interesting stories regarding symmetric inequalities that will be unveiled in the following chapters; right now, in these pages, we will review three essential matters: primary symmetric polynomials, normalization skill and symmetric separation.

8.1 Getting started

In general, a symmetric inequality of n variables $a_1, a_2, ..., a_n$ can be rewritten as

$$f(a_1, a_2, ..., a_n) \geq 0$$

where

$$f(a_1, a_2, ..., a_n) = f\left(a_{i_1}, a_{i_2}, ..., a_{i_n}\right)$$

for all permutations $(i_1, i_2, ..., i_n)$ of $(1, 2, ..., n)$.

Because of the symmetry, we can rearrange the order of variables (that means we can choose an arbitrary order). Because of the symmetry, we can estimate a mixed expression by smaller expressions of one-variable.

Schur inequality is a very important symmetric inequality and it would be a shortcoming if it wasn't discussed now.

Theorem 10 (Schur inequality). *Suppose that a, b, c are non-negative real numbers, then*

$$a^3+b^3+c^3+3abc \geq ab(a+b)+bc(b+c)+ca(c+a).$$

Proof. Because of the symmetry, we can assume that $a \geq b \geq c$. Let $x=a-b, y=b-c$ and rewrite the inequality to the following form

$$\sum_{cyc} a(a-b)(a-c) \geq 0 \Leftrightarrow c(x+y)y-(c+y)xy+(c+x+y)x(x+y) \geq 0$$

$$\Leftrightarrow c(x^2+xy+y^2)+x^2(x+2y) \geq 0$$

which is obvious because $c, x, y \geq 0$. The equality holds for $x=y=0$ and $x=c=0$, that means $a=b=c$ or $a=b, c=0$ up to permutation.

Comment. This inequality is, in fact, equivalent to the following well-known result

★ *Suppose that a, b, c are non-negative real numbers. Prove that*

$$(a+b-c)(b+c-a)(c+a-b) \leq abc.$$

Naturally, we may wonder if the following similar result is true or false

★ *Let a, b, c be positive real numbers. Prove or disprove that*

$$a^6+b^6+c^6+3a^2b^2c^2 \geq a^5(b+c)+b^5(c+a)+c^5(a+b).$$

Unfortunately, this inequality is wrong. We only need to choose $a \to 0$ and $b=c$. There is even no positive constant k such that

$$a^6+b^6+c^6+ka^2b^2c^2 \geq a^5(b+c)+b^5(c+a)+c^5(a+b).$$

However, the following inequality holds

★ *Let a, b, c be three real numbers. Prove that*

$$a^6+b^6+c^6+a^2b^2c^2 \geq \frac{2}{3}\left(a^5(b+c)+b^5(c+a)+c^5(a+b)\right).$$

Solution. According to **AM-GM** inequality and **Schur** inequality, we deduce that

$$3\sum_{cyc} a^6+3a^2b^2c^2 \geq 2\sum_{cyc} a^6+\sum_{cyc} a^4(b^2+c^2)$$

$$=\sum_{cyc}(a^6+a^4b^2)+\sum_{cyc}(a^6+a^4c^2) \geq 2\sum_{cyc} a^5(b+c).$$

∇

Theorem 11 (General Schur inequality). *Suppose that a, b, c are non-negative real numbers and k is a positive constant, then*

$$a^k(a-b)(a-c)+b^k(b-a)(b-c)+c^k(c-a)(c-b)\geq 0.$$

PROOF. Certainly, we may assume that $a \geq b \geq c$. In this case, we have

$$c^k(c-a)(c-b)\geq 0,$$

$$a^k(a-b)(a-c)+b^k(b-a)(b-c)=(a-b)\left((a^{k+1}-b^{k+1})+c(a^k-b^k)\right)\geq 0.$$

Summing up these results, we are done. The equality holds for $a=b=c$ and $a=b, c=0$ up to permutation.

Comment. By a similar approach, we can prove that the inequality is still true if $k \leq 0$. Morever, if k is an even integer, the inequality is true for all real numbers a, b, c (not necessarily positive).

∇

Example 8.1.1. *Let a, b, c be non-negative real numbers with sum 2. Prove that*

$$a^4+b^4+c^4+abc\geq a^3+b^3+c^3.$$

SOLUTION. According to the fourth degree-**Schur** inequality, we have

$$a^4+b^4+c^4+abc(a+b+c)\geq a^3(b+c)+b^3(c+a)+c^3(a+b)$$

or equivalently

$$2(a^4+b^4+c^4)+abc(a+b+c)\geq (a^3+b^3+c^3)(a+b+c).$$

Putting $a+b+c=2$ into the last inequality, we get the desired result. Equality holds for $a=b=c=\frac{2}{3}$, or $a=b=1, c=0$ or permutations.

∇

Example 8.1.2. *Let a, b, c be positive real numbers. Prove that*

$$\frac{a^2}{\sqrt{(b+c)(b^3+c^3)}}+\frac{b^2}{\sqrt{(c+a)(c^3+a^3)}}+\frac{c^2}{\sqrt{(a+b)(a^3+b^3)}}\geq\frac{3}{2}.$$

(Pham Kim Hung)

SOLUTION. By **Hölder** inequality, we deduce that

$$\left(\sum_{cyc}\frac{a^2}{\sqrt{(b+c)(b^3+c^3)}}\right)^2\left(\sum_{cyc}a^2(b+c)(b^3+c^3)\right)\geq\left(\sum_{cyc}a^2\right)^3.$$

So it is enough to prove that

$$4\left(\sum_{cyc}a^2\right)^3\geq 9\sum_{cyc}a^2(b+c)(b^3+c^3)$$

$$\Leftrightarrow\ 4\sum_{cyc}a^6+3\sum_{cyc}a^4(b^2+c^2)+24a^2b^2c^2\geq 9abc\sum_{cyc}a^2(b+c).$$

According to the third degree-**Schur** inequality $\sum_{cyc}a^2(b+c)\leq\sum_{cyc}a^3+3abc$, so it's enough to prove that

$$4\sum_{cyc}a^6+3\sum_{cyc}a^4(b^2+c^2)\geq 9\sum_{cyc}a^4bc+3a^2b^2c^2$$

which is obvious by **AM-GM** inequality because

$$2\sum_{cyc}a^6=\sum_{cyc}(a^6+b^6)\geq\sum_{cyc}a^2b^2(a^2+b^2)=\sum_{cyc}a^4(b^2+c^2)\geq 2\sum_{cyc}a^4bc\geq 6a^2b^2c^2.$$

We are done. Equality holds for $a=b=c$.

∇

Example 8.1.3. *Suppose that a,b,c are non-negative real numbers. Prove that*

$$\frac{a^2}{2b^2-bc+2c^2}+\frac{b^2}{2c^2-ca+2a^2}+\frac{c^2}{2a^2-ab+2b^2}\geq 1.$$

(Vasile Cirtoaje)

SOLUTION. According to **Cauchy-Schwarz** inequality, we deduce that

$$\sum_{cyc}\frac{a^2}{2b^2-bc+2c^2}\geq\frac{(a^2+b^2+c^2)^2}{a^2(2b^2-bc+2c^2)+b^2(2c^2-ca+2a^2)+c^2(2a^2-ac+2b^2)}.$$

It suffices to prove that

$$\left(\sum_{cyc}a^2\right)^2\geq\sum_{cyc}a^2(2b^2-bc+2c^2)\ \Leftrightarrow\ \sum_{cyc}a^4+abc\left(\sum_{cyc}a\right)\geq 2\sum_{cyc}a^2b^2.$$

According to the fourth degree-**Schur** inequality, we conclude that

$$\sum_{cyc}a^4+abc\left(\sum_{cyc}a\right)\geq\sum_{cyc}ab(a^2+b^2)\geq 2\sum_{cyc}a^2b^2.$$

We are done. Equality holds for $a=b=c$ and $a=b,c=0$ or permutations.

∇

Example 8.1.4. *Let a, b, c be non-negative real numbers. Prove that*

$$\frac{a^3}{b^2-bc+c^2}+\frac{b^3}{c^2-ca+a^2}+\frac{c^3}{a^2-ab+b^2}\ge a+b+c.$$

SOLUTION. Applying **Cauchy-Schwarz** inequality, we have

$$\sum_{cyc}\frac{a^3}{b^2-bc+c^2}=\sum_{cyc}\frac{a^4}{a(b^2-bc+c^2)}\ge\frac{(a^2+b^2+c^2)^2}{\sum_{cyc}a(b^2-bc+c^2)}.$$

It remains to prove that

$$\left(\sum_{cyc}a^2\right)^2\ge\left(\sum_{cyc}a(b^2-bc+c^2)\right)\left(\sum_{cyc}a\right)$$

or

$$\sum_{cyc}a^4+2\sum_{cyc}a^2b^2\ge(a+b+c)\sum_{cyc}a^2(b+c)-3abc\sum_{cyc}a$$

or

$$\sum_{cyc}a^4+abc\sum_{cyc}a\ge\sum_{cyc}a^3(b+c).$$

This is exacly the fourth degree-**Schur** inequality, so we are done. Equality holds for $a=b=c$ or $a=b, c=0$ up to permutation.

∇

Example 8.1.5. *Let a, b, c be non-negative real numbers. Prove that*

$$a^2\sqrt{b^2-bc+c^2}+b^2\sqrt{c^2-ca+a^2}+c^2\sqrt{a^2-ab+b^2}\le a^3+b^3+c^3.$$

SOLUTION. Applying **AM-GM** inequality, we have

$$\sum_{cyc}a^2\sqrt{b^2-bc+c^2}=a\sqrt{a^2(b^2-bc+c^2)}\le\frac{1}{2}\sum_{cyc}a(a^2+b^2+c^2-bc).$$

Then, by the third degree-**Schur** inequality we get that

$$2\sum_{cyc}a^3-\sum_{cyc}a(a^2+b^2+c^2-bc)=\sum_{cyc}a^3+3abc-\sum_{cyc}ab(a+b)\ge 0.$$

We are done. Equality holds for $a=b=c$ or $a=b, c=0$ up to permutation.

∇

Example 8.1.6. *Let a, b, c be non-negative real numbers. Prove that*

$$\frac{a^3}{\sqrt{b^2-bc+c^2}}+\frac{b^3}{\sqrt{c^2-ca+a^2}}+\frac{c^3}{\sqrt{a^2-ab+b^2}}\ge a^2+b^2+c^2.$$

(Vo Quoc Ba Can)

SOLUTION. Applying **Cauchy-Schwarz** inequality, we deduce that

$$\sum_{cyc} \frac{a^3}{\sqrt{b^2 - bc + c^2}} \ge \frac{(a^2+b^2+c^2)^2}{a\sqrt{b^2-bc+c^2} + b\sqrt{c^2-ca+a^2} + c\sqrt{a^2-ab+b^2}}.$$

So it is enough to prove that

$$\sum_{cyc} a\sqrt{b^2-bc+c^2} \le a^2+b^2+c^2.$$

Cauchy-Schwarz inequality yields that

$$\left(\sum_{cyc} a\sqrt{b^2-bc+c^2}\right)^2 \le \left(\sum_{cyc} a\right)\left(\sum_{cyc} a(b^2-bc+c^2)\right).$$

Thwn, by **Schur** inequality we deduce that

$$\left(\sum_{cyc} a^2\right)^2 - \left(\sum_{cyc} a\right)\left(\sum_{cyc} a(b^2-bc+c^2)\right) = \sum_{cyc} a^4 + abc\sum_{cyc} a - \sum_{cyc} a^3(b+c) \ge 0.$$

The proof is finished. Equality holds for $a=b=c$ and $a=b, c=0$ and permutations.

∇

8.2 Primary symmetric polynomials

Suppose that $x_1, x_2, ..., x_n$ are real numbers. We define their primary symmetric polynomials for each $k \in \{1, 2, ..., n\}$ as follow

$$S_k = \sum x_{i_1} x_{i_2} ... x_{i_k}$$

in which the sum is taken over all $\{i_1, i_2, ..., i_k\} \subset \{1, 2, ..., n\}$.

An important and classical result on primary symmetric polynomials is that

★ *Every symmetric polynomial of $x_1, x_2, ..., x_n$ can be expressed as a polynomial with variables the primary symmetric polynomials of $x_1, x_2, ..., x_n$.*

The proof of this theorem won't be showed here and it would better be solved by yourself as an algebraic exercise. According to this theorem, examining symmetric expressions can be turned into examining primary symmetric polynomials. In this part, however, we only concentrate on applying primary symmetric polynomials in proving three-variable inequalities.

Example 8.2.1. *Let a, b, c be positive real numbers with sum 2. Prove that*

$$a^2b^2 + b^2c^2 + c^2a^2 + abc \le 1.$$

(Pham Kim Hung)

SOLUTION. Because $a+b+c=2$, the inequality is equivalent to

$$(ab+bc+ca)^2 \le 1+3abc.$$

Denote $x = ab+bc+ca$ and $y = abc$. We are done if $x \le 1$. Otherwise, $x \ge 1$, and by AM-GM inequality, we deduce that

$$\prod_{cyc}(a+b-c) \le \prod abc \Rightarrow 8\prod_{cyc}(1-a) \le \prod_{cyc} abc \Rightarrow 8+9y \ge 8x.$$

So it suffices to prove that

$$x^2 \le 1 + \frac{1}{3}(8x-8) \Leftrightarrow 3x^2 - 8x + 5 \le 0 \Leftrightarrow (x-1)(3x-5) \le 0$$

which is obvious because $1 \le x \le \frac{4}{3} < \frac{5}{3}$. The equality holds for $a=b=1, c=0$ up to permutation.

∇

Example 8.2.2. *Suppose that a, b, c are non-negative real numbers such that $a^2 + b^2 + c^2 = 1$. Prove the inequality*

$$a+b+c \le \sqrt{2} + \frac{9abc}{4}.$$

SOLUTION. We denote $x = a+b+c$, $y = ab+bc+ca$ and $z = abc$. By the fourth degree-Schur inequality, we have

$$\sum_{cyc} a^4 + abc\sum_{cyc} a \ge \sum_{cyc} a^3(b+c)$$

$$\Leftrightarrow \left(\sum_{cyc} a^2\right)^2 - 2\left(\sum_{cyc} ab\right)^2 + 6abc\left(\sum_{cyc} a\right) \ge \left(\sum_{cyc} a^2\right)\left(\sum_{cyc} ab\right)$$

$$\Leftrightarrow 1 - 2y^2 + 6xz \ge y \Leftrightarrow z \ge \frac{2y^2+y-1}{6x} \quad (\star)$$

Notice that $x = \sqrt{1+2y}$, so if $y \le \frac{1}{2}$, we have done (because $x \le \sqrt{2}$). Otherwise, according to $(\star)$, it suffices to prove that

$$\sqrt{1+2y} \le \sqrt{2} + \frac{9(2y^2+y-1)}{24\sqrt{1+2y}}$$

$$\Leftrightarrow\ (2y-1)\frac{\sqrt{1+2y}}{\sqrt{2}+\sqrt{1+2y}} \le \frac{9(2y-1)(y+1)}{24}.$$

Because $1 \ge y \ge \frac{1}{2}$, we conclude that

$$\frac{\sqrt{1+2y}}{\sqrt{2}+\sqrt{1+2y}} \le \frac{\sqrt{3}}{\sqrt{3}+\sqrt{2}} < \frac{9}{16} \le \frac{9(y+1)}{24}.$$

Equality holds for $a = b = \frac{1}{\sqrt{2}}, c = 0$ or permutations.

∇

Example 8.2.3. *Let a, b, c be positive real numbers satisfying $abc = 1$. Prove that*

$$\frac{1}{1+a+b}+\frac{1}{1+b+c}+\frac{1}{1+c+a} \le \frac{1}{2+a}+\frac{1}{2+b}+\frac{1}{2+c}.$$

(Bulgarian MO 1998)

SOLUTION. Denote $S = \sum_{cyc} a, P = \sum_{cyc} ab, Q = abc$. By some calculations, we get that

$$\text{LHS} = \sum_{cyc} \frac{1}{S+1-a} = \frac{S^2+4S+3+P}{S^2+2S+PS+P},$$

$$\text{RHS} = \sum_{cyc} \frac{1}{2+a} = \frac{12+4S+P}{9+4S+2P}.$$

So it suffices to prove that

$$\frac{S^2+4S+3+P}{S^2+2S+PS+P} \le \frac{12+4S+P}{9+4S+2P},$$

which is reduced to

$$(3P-5)S^2+(S-1)P^2+6PS \ge 24S+3P+27.$$

Because $abc = 1$, we deduce $S, P \ge 3$, therefore

$$\text{LHS} \ge 4S^2+2P^2+6PS \ge 12S+6(P-1)S+6S+2P^2$$
$$\ge 24S+3P+(P^2+6S) \ge \text{RHS}.$$

Equality holds for $S = P = 3$ or equivalently to $a = b = c = 1$.

∇

Example 8.2.4. *Let a, b, c be positive real numbers. Prove that*

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+\frac{abc}{2(a^3+b^3+c^3)} \ge \frac{5}{3}.$$

(Pham Kim Hung)

SOLUTION. WLOG, we may assume that $a+b+c=3$. Denote x$=ab+bc+ca$ and $y=abc$. Then we have

$$\frac{abc}{a^3+b^3+c^3}=\frac{y}{27+3y-9x},$$
$$\sum_{cyc}\frac{a}{b+c}=\frac{27+3y-6x}{3x-y}.$$

We need to prove that

$$\frac{27+3y-6x}{3x-y}+\frac{y}{2(27+3y-9x)}\ge\frac{5}{3}.$$

By **AM-GM** inequality, $\prod_{cyc}(3-2a)\le\prod_{cyc}a$, so $9+3y\ge 4x$. Moreover, the left hand side of the above expression is a strictly increasing function of y, so it suffices to prove that

$$\frac{27+(4x-9)-6x}{3x-\frac{1}{3}(4x-9)}+\frac{(4x-9)}{6(27+(4x-9)-9x)}\ge\frac{5}{3}$$

$$\Leftrightarrow\ \frac{3(18-2x)}{9+5x}+\frac{4x-9}{6(18-5x)}\ge\frac{5}{3}\ \Leftrightarrow\ \frac{3(3-x)(153-50x)}{2(9+5x)(18-5x)}\ge 0,$$

which is obvious because $x\le 3$. Equality holds for $x=3$ or $a=b=c=1$.

∇

Example 8.2.5. *Let a,b,c be real numbers with sum 3. Prove that*

$$\left(1+a+a^2\right)\left(1+b+b^2\right)\left(1+c+c^2\right)\ge 9(ab+bc+ca).$$

(Pham Kim Hung)

SOLUTION. We denote

$$x=a+b+c,\ y=ab+bc+ca,\ z=abc.$$

According to the hypothesis, $x=3$, so we can rewrite the inequality to

$$z^2+z+1+\sum_{sym}(a+a^2)+\sum_{sym}ab+\sum_{sym}a^2b^2+abc\left(\sum_{sym}a+\sum_{sym}ab\right)+\sum_{sym}a^2(b+c)\ge 9y$$

$$\Leftrightarrow\ z^2+z+1+x+(x^2-2y)+y+(y^2-2xz)+z(x+y)+xy-3z\ge 9y$$

$$\Leftrightarrow\ z^2-2z+1+(x-y)-z(x-y)+(x-y)^2+3xy\ge 9y$$

$$\Leftrightarrow\ (z-1)^2-(z-1)(x-y)+(x-y)^2\ge 0.$$

The last inequality is obvious. Equality holds for $z=1, x=y$ or $a=b=c=1$.

∇

Example 8.2.6. *Let a, b, c be positive real numbers such that $abc = 1$. Prove that*

$$\frac{1}{1+3a} + \frac{1}{1+3b} + \frac{1}{1+3c} + \frac{1}{1+a+b+c} \geq 1.$$

(Pham Kim Hung)

SOLUTION. We denote $x = a+b+c$ and $y = ab+bc+ca$. Then the inequality can be rewritten to

$$\frac{3+6x+9y}{28+3x+9y} + \frac{1}{1+x} \geq 1 \Leftrightarrow \frac{1}{1+x} \geq \frac{25-3x}{28+3x+9y}$$

$$\Leftrightarrow 3x^2 - 19x + 9y + 3 \geq 0.$$

Denote $z = \sqrt{\frac{x}{3}}$. Because $y^2 = (ab+bc+ca)^2 \geq 3abc(a+b+c) = 9z^2$, it follows that $y \geq 3z$. Therefore it suffices to prove that

$$27z^4 - 57z^2 + 27z + 3 \geq 0 \Leftrightarrow 3(z-1)(9z^3 + 9z^2 - 10z - 1) \geq 0,$$

which is obviously true because $z \geq 1$. Clearly, the equality holds for $a = b = c = 1$.

∇

Example 8.2.7. *Let a, b, c be non-negative real numbers with sum 1. Prove that*

$$\frac{ab+bc+ca}{a^2+b^2+c^2+16abc} \geq 8(a^2b^2+b^2c^2+c^2a^2).$$

(Pham Kim Hung, MYM)

SOLUTION. Denote $x = 4(ab+bc+ca)$ and $y = 8abc$ then we obtain

$$a^2+b^2+c^2 = 1 - \frac{x}{2} \ ; \ a^2b^2+b^2c^2+c^2a^2 = \frac{x^2}{16} - \frac{y}{4}.$$

We can rewrite the inequality to the form

$$2x \geq (4-2x+8y)(x^2-4y)$$

$$\Leftrightarrow x(x-1)^2 \geq 4y\left((x-1)(x+2) - 4y\right)$$

$$\Leftrightarrow x(x-1)^2 + 16y^2 \geq 4y(x-1)(x+2).$$

Obviously, $x \leq \frac{4}{3}$. If $x \leq 1$, we are done immediately. Otherwise, suppose that $x \geq 1$. By the third degree-Schur inequality, it's easy to get $8(x-1) \leq 9y$. Considering x as a parameter in $\left[1, \frac{4}{3}\right]$, we will prove that $f(y) \geq 0$, where

$$f(y) = 16y^2 - 4y(x-1)(x+2) + x(x-1)^2.$$

Indeed, notice that $x \geq 1$, so $f(y)$ is an increasing function because

$$f'(y) = 32y - 4(x-1)(x+2) \geq \frac{32 \cdot 8(x-1)}{9} - 4(x-1)(x+2)$$
$$\geq \frac{256(x-1)}{9} - \frac{40(x-1)}{3} \geq 0.$$

Therefore, it's sufficient to prove that $f\left(\frac{8(x-1)}{9}\right) \geq 0$ or

$$16\cdot\left(\frac{8(x-1)}{9}\right)^2 - 4\left(\frac{8(x-1)}{9}\right)(x-1)(x+2) + x(x-1)^2 \geq 0$$

$$\Leftrightarrow \frac{1024}{81} - \frac{32}{9}(x+2) + x \geq 0 \Leftrightarrow \frac{448}{81} \geq \frac{23x}{9},$$

which is true because $x \leq \frac{4}{3}$. The problem has been completely solved and equality holds for $a = b = \frac{1}{2}, c = 0$ or permutations.

Comment. We have a similar inequality as follows

★ *Let a, b, c be three side lengths of a triangle whose perimeter is 1. Prove that*

$$\frac{ab+bc+ca}{a^2b^2+b^2c^2+c^2a^2+\frac{19}{8}abc} \leq 8(a^2+b^2+c^2).$$

∇

Example 8.2.8. *Let a, b, c be non-negative real numbers with sum 2. Prove that*

$$a^2+b^2+c^2 \geq 2(a^3b^3+b^3c^3+c^3a^3+4a^2b^2c^2).$$

(Pham Kim Hung)

SOLUTION. Write $p = ab+bc+ca$ and $q = abc$. The inequality can be rewritten to

$$(a+b+c)^2-2(ab+bc+ca) \geq 2(ab+bc+ca)^3-6abc(a+b+c)(ab+bc+ca)+14a^2b^2c^2.$$

$$\Leftrightarrow 2-p \geq p^3-6pq+7q^2.$$

Let $r = \max\left\{0, \frac{8p-8}{9}\right\}$, then $q \geq r$ according to **Schur** inequality. Consider the function

$$f(q) = 7q^2 - 6pq + p^3 + p - 2.$$

Since $f'(q) = 14q - 6p = 14abc - 3(a+b+c)(ab+bc+ca) < 0$, we deduce that $f(q) \leq f(r)$. If $p \leq 1$ then $r = 0$ and we can conclude that

$$f(q) \leq f(r) = p^3 + p - 2 = (p-1)(p^2+p+2) \leq 0.$$

If $p \geq 1$, we get $r = \dfrac{8(p-1)}{9}$, and the inequality $f(r) \leq 0$ is equivalent to

$$7\left(\frac{8(p-1)}{9}\right)^2 - 6p\left(\frac{8(p-1)}{9}\right) + (p-1)(p^2+p+2) \leq 0$$

$$\Leftrightarrow \frac{448(p-1)}{81} - \frac{16p}{3} + p^2 + p + 2 \leq 0 \Leftrightarrow p^2 + \frac{37p}{81} - \frac{236}{81} \leq 0.$$

This last inequality is true since $p \leq \frac{4}{3}$. Equality holds for $a = b = 1, c = 0$ or permutations.

∇

8.3 Normalization skill

An important technique that is frequently used in proving symmetric inequalities is normalization. To understand this technique, we first need to clarify the difference between homogeneous functions and non-homogeneous functions.

Definition 2. *Suppose that f is a function of n variables $a_1, a_2, ..., a_n$. We say that f is a homogeneous function if and only if there exist a real number k such that*

$$f(ta_1, ta_2, ..., ta_n) = t^k f(a_1, a_2, ..., a_n) \; \forall t, a_1, a_2, ..., a_n \in R.$$

Almost all inequalities we have seen so far are homogeneous. In this case, a condition between variables $x_1, x_2, ..., x_n$ such as $x_1 + x_2 + ... + x_n = n$ or $x_1x_2...x_n = 1$ is meaningless (because we can divide (or multiply) each variable by arbitrary real numbers but the result of the problem is not affected). Sometimes, the condition only helps simplify the appearance of the problem, as with the following example

Example 8.3.1. *Suppose that a, b, c are three real numbers satisfying $a^2+b^2+c^2 = 3$. Prove the following inequality*

$$a^3(b+c) + b^3(c+a) + c^3(a+b) \leq 6.$$

SOLUTION. Certainly, the inequality is non-homogeneous. However, the condition $a^2 + b^2 + c^2 = 3$ can help change it to homogeneous as

$$a^3(b+c) + b^3(c+a) + c^3(a+b) \leq \frac{2}{3}(a^2+b^2+c^2)^2.$$

Rewrite this inequality to

$$2\sum_{cyc} a^4 + 4\sum_{cyc} a^2b^2 \geq 3\sum_{cyc} ab(a^2+b^2)$$

$$\Leftrightarrow \sum_{cyc}\left(a^4+b^4-3ab(a+b)+4a^2b^2\right)\geq 0 \Leftrightarrow \sum_{cyc}(a-b)^4+3\sum_{cyc}ab(a-b)^2\geq 0,$$

which is obvious. Equality holds for $a=b=c$, and then $a=b=c=1$.

Comment. Consider the general problem as follows

★ *Suppose that $a_1,a_2,...,a_n$ are non-negative real numbers such that $a_1^2+a_2^2+$ $+...+a_n^2=n$. For what value of n then the following inequality holds*

$$a_1^3(a_2+a_3+...+a_n)+a_2^3(a_1+a_3+...+a_n)+...+a_n^3(a_1+a_2+...+a_{n-1})\leq n(n-1)?$$

Only two numbers satisfy this condition: $n=3$ and $n=4$. If $n=4$, the inequality is (after changing a_1,a_2,a_3,a_4 to a,b,c,d)

$$4\sum_{cyc}a^3(b+c+d)\leq 3(a^2+b^2+c^2+d^2)^2$$

$$\Leftrightarrow \sum_{cyc}(a^6+b^4-4a^3b-4b^3a+6a^2b^2)\geq 0 \Leftrightarrow \sum_{cyc}(a-b)^4\geq 0.$$

∇

Changing non-homogeneous inequalities to homogeneous inequalities as above seems to be very intuitive. But what about the reverse? Is it unreasonable if we change a homogeneous inequality to a non-homogeneous one? Does it have any meaning? To answer this question, let's see an example

Example 8.3.2. *Let a,b,c be non-negative real numbers. Prove that*

$$\sqrt{\frac{ab+bc+ca}{3}}\leq\sqrt[3]{\frac{(a+b)(b+c)(c+a)}{8}}.$$

Solution. WLOG, suppose that $ab+bc+ca=3$. By **AM-GM** inequality, we deduce that $a+b+c\geq 3$ and $abc\leq 1$. Therefore

$$(a+b)(b+c)(c+a)=(a+b+c)(ab+bc+ca)-abc=3(a+b+c)-abc\geq 8$$

$$\Rightarrow \sqrt{\frac{ab+bc+ca}{3}}\leq 1\leq\sqrt[3]{\frac{(a+b)(b+c)(c+a)}{8}}.$$

The proof is completed. Equality holds for $a=b=c$.

∇

Let's review this solution. Its exceptional feature is the step of assuming that $ab+bc+ca=3$. Why can we do so? In fact, if $a=b=c=0$, the inequality

is obvious. Otherwise, let $a' = \frac{a}{t}, b' = \frac{b}{t}, c' = \frac{c}{t}$ $(t > 0)$. The inequality is true for a, b, c if and only if it's true for a', b', c'. Just choose $t = \sqrt{\frac{ab+bc+ca}{3}}$ then $a'b' + b'c' + c'a' = 3$. Because the inequality is true for a', b', c' (as we proved), it must be true for a, b, c.

Let's analyze another fact. What happen if we suppose that $a + b + c = 3$ or $abc = 1$ instead of $ab + bc + ca = 3$? However, they do bring us either a much more complicated solution or even nothing.

The procedure we used is called normalization. This skill is widely applied for homogeneous inequalities because these inequalities allows us to suppose anything that we need: $a+b+c = 3, ab+bc+ca = 3$, etc. Sometimes, solutions by normalization are unexpectedly short and nice as in the following examples

Example 8.3.3. *Let a, b, c be non-negative real numbers. Prove that*

$$\frac{(2a+b+c)^2}{2a^2+(b+c)^2} + \frac{(2b+c+a)^2}{2b^2+(c+a)^2} + \frac{(2c+a+b)^2}{2c^2+(a+b)^2} \le 8.$$

(USA MO 2003)

SOLUTION. By normalizing the expression with $a + b + c = 3$, we reduce the left expression to a simpler form

$$\frac{(3+a)^2}{2a^2+(3-a)^2} + \frac{(3+b)^2}{2b^2+(3-b)^2} + \frac{(3+c)^2}{2c^2+(3-c)^2}.$$

Notice that

$$\frac{3(3+a)^2}{2a^2+(3-a)^2} = \frac{a^2+6a+9}{a^2-2a+3} = 1 + \frac{8a+6}{(a-1)^2+2}$$
$$\le 1 + \frac{8a+6}{2} = 4a+4.$$

We conclude that

$$\sum_{cyc} \frac{(3+a)^2}{2a^2+(3-a)^2} \le \frac{1}{3}\left(12 + 4\sum_{cyc} a\right) = 8.$$

∇

Example 8.3.4. *Let a, b, c be non-negative real numbers. Prove that*

$$\frac{(b+c-a)^2}{(b+c)^2+a^2} + \frac{(c+a-b)^2}{(c+a)^2+b^2} + \frac{(a+b-c)^2}{(a+b)^2+c^2} \ge \frac{3}{5}.$$

(Japan MO 2002)

SOLUTION. WLOG, we may suppose that $a+b+c=3$. The inequality becomes

$$\sum_{cyc}\frac{(3-2a)^2}{a^2+(3-a)^2}\geq\frac{3}{5}\Leftrightarrow\sum_{cyc}\frac{1}{2a^2-6a+9}\leq\frac{3}{5}.$$

Just notice that

$$\sum_{cyc}\left(\frac{5}{2a^2-6a+9}-1\right)=\sum_{cyc}\frac{2(a-1)(a-2)}{2a^2-6a+9}$$

$$=\sum_{cyc}\left(\frac{-2(a-1)}{5}+\frac{(a-1)^2(2a+1)}{5(2a^2-6a+9)}\right)\geq\sum_{cyc}\frac{-2(a-1)}{5}=0.$$

∇

Example 8.3.5. *Let a,b,c be positive real numbers. Prove that*

$$\frac{(2a+b+c)^2}{4a^3+(b+c)^3}+\frac{(2b+c+a)^2}{4b^3+(c+a)^3}+\frac{(2c+a+b)^2}{4c^3+(a+b)^3}\leq\frac{12}{a+b+c}.$$

(Pham Kim Hung)

SOLUTION. Suppose that $a+b+c=3$. The problem becomes

$$\sum_{cyc}\frac{(3+a)^2}{4a^3+(3-a)^3}\leq 4.$$

Notice that

$$\frac{(3+a)^2}{4a^3+(3-a)^3}-\frac{4}{3}=\frac{(a-1)(-4a^2-15a+27)}{4a^3+(3-a)^3}.$$

$$=(a-1)\left(\frac{2}{3}+\frac{(a-1)(-2a^2-12a-9)}{4a^3+(3-a)^3}\right)\leq\frac{2(a-1)}{3}.$$

We conclude that

$$\sum_{cyc}\frac{(3+a)^2}{4a^3+(3-a)^3}\leq\sum_{cyc}\left(\frac{4}{3}+\frac{2(a-1)}{3}\right)=4.$$

∇

Example 8.3.6. *Let a,b,c,d be non-negative real numbers. Prove that*

$$\frac{a}{b^2+c^2+d^2}+\frac{b}{c^2+d^2+a^2}+\frac{c}{d^2+a^2+b^2}+\frac{d}{a^2+b^2+c^2}\geq\frac{3\sqrt{3}}{2}\cdot\frac{1}{\sqrt{a^2+b^2+c^2+d^2}}.$$

SOLUTION. WLOG, assume that $a^2+b^2+c^2+d^2=1$. The problem becomes

$$\frac{a}{1-a^2}+\frac{b}{1-b^2}+\frac{c}{1-c^2}+\frac{d}{1-d^2}\ge\frac{3\sqrt{3}}{2}.$$

By **AM-GM** inequality, we get

$$2a^2(1-a^2)(1-a^2)\le\left(\frac{2}{3}\right)^3\Rightarrow a(1-a^2)\le\frac{2}{3\sqrt{3}}\Rightarrow\frac{a}{1-a^2}\ge\frac{3\sqrt{3}}{2}a^2.$$

Therefore we conclude that

$$\sum_{cyc}\frac{a}{1-a^2}\ge\frac{3\sqrt{3}}{2}\left(\sum_{cyc}a^2\right)=\frac{3\sqrt{3}}{2}.$$

Equality holds for $a=b=c, d=0$ up to permutation.

∇

Example 8.3.7. *Let a,b,c be non-negative real numbers. Prove that*

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+\frac{3\sqrt[3]{abc}}{2(a+b+c)}\ge 2.$$

(Pham Kim Hung)

SOLUTION. Applying **Cauchy-Schwarz** inequality, we get

$$\sum_{cyc}\frac{a}{b+c}\ge\frac{(a+b+c)^2}{2(ab+bc+ca)}.$$

We normalize $a+b+c=1$ and prove that

$$\frac{1}{2x}+\frac{3}{2}\sqrt[3]{abc}\ge 2,$$

where $x=ab+bc+ca\le\frac{1}{3}$. If $x\le\frac{1}{4}$, we are done immediately. Otherwise, by **Schur** inequality, we have $9abc\ge 4x-1\ge 0$, so it suffices to prove that

$$\frac{1}{2x}+\frac{\sqrt[3]{3}}{2}\sqrt[3]{4x-1}\ge 2,$$

or

$$3x^3(4x-1)\ge(4x-1)^3,$$

or

$$(4x-1)(3x-1)(x^2-5x+1)\ge 0\ ;$$

This last inequality is true since $\frac{1}{4}\le x\le\frac{1}{3}$ (therefore $x^2-5x+1<0$), and the desired result follows. Equality holds for $a=b=c$ or $a=b, c=0$.

∇

Example 8.3.8. *Let a, b, c be non-negative real numbers. Prove that*

$$\frac{1}{(a+b)^2}+\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}\geq\frac{9}{4(ab+bc+ca)}.$$

(Iran TST 1996)

SOLUTION. We normalize $ab+bc+ca=1$. The inequality becomes

$$4\sum_{cyc}(a+b)^2(a+c)^2\geq 9(a+b)^2(b+c)^2(c+a)^2$$

or

$$4(1+a^2)^2+4(1+b^2)^2+4(1+c^2)^2\geq 9(a+b+c-abc)^2.$$

Denote $s=a+b+c$. We can rewrite the inequality in terms of s and abc as follows

$$4(s^4-2s^2+1+4sabc)\geq 9(s-abc)^2.$$

If $s\geq 2$, we get the conclusion immediately because

$$\text{LHS}\geq 4(s^4-2s^2+1)=9s^2+(s^2-4)(4s^2-1)\geq 9s^2\geq 9(s-abc)^2=\text{RHS}.$$

Otherwise, we may assume that $s\leq 2$. According to **Schur** inequality, we get

$$\sum_{cyc}a^4+abc\sum_{cyc}a\geq\sum_{cyc}a^3(b+c)\ \Rightarrow 6abcs\geq(4-s^2)(s^2-1).$$

Moreover, $9abc\leq(a+b+c)(ab+bc+ca)=s$, so we conclude that

$$\text{LHS}-\text{RHS}=(s^2-4)(4s^2-1)+34sabc-9a^2b^2c^2\geq(s^2-4)(4s^2-1)+33sabc$$

$$\geq(s^2-4)(4s^2-1)+\frac{11}{2}(4-s^2)(s^2-1)=\frac{3}{2}(4-s^2)(s^2-3)\geq 0.$$

This ends the proof. Equality holds for $a=b=c$ or $(a,b,c)=(1,1,0)$.

∇

Example 8.3.9. *Let a, b, c, d be positive real numbers. Prove that*

$$\frac{abc}{(d+a)(d+b)(d+c)}+\frac{bcd}{(a+b)(a+c)(a+d)}+$$

$$+\frac{cda}{(b+a)(b+c)(b+d)}+\frac{dab}{(c+a)(c+b)(c+d)}\geq\frac{1}{2}.$$

(Nguyen Van Thach)

SOLUTION. Denote $x = \frac{1}{a}, y = \frac{1}{b}, z = \frac{1}{c}, t = \frac{1}{d}$, the inequality becomes

$$\frac{x^3}{(x+y)(x+z)(x+t)} + \frac{y^3}{(y+x)(y+z)(y+t)} +$$
$$+\frac{z^3}{(z+x)(z+y)(z+t)} + \frac{t^3}{(t+x)(t+y)(t+z)} \geq \frac{1}{2}.$$

WLOG, assume that $x+y+z+t=4$. By **AM-GM** inequality, we have

$$(x+y)(x+z)(x+t) \leq \left(x + \frac{y+z+t}{3}\right)^3 = \left(x + \frac{4-x}{3}\right)^3 = \frac{8}{27}(x+2)^3.$$

So we only need to prove that

$$\frac{x^3}{(x+2)^3} + \frac{y^3}{(y+2)^3} + \frac{z^3}{(z+2)^3} + \frac{t^3}{(t+2)^3} \geq \frac{4}{27}.$$

But, it is easy to check that

$$\frac{x^3}{(x+2)^3} - \frac{2x-1}{27} = \frac{2(x-1)^2(-x^2+6x+4)}{27(x+2)^2} \geq 0$$

because $0 \leq x \leq 4$. We can conclude that

$$\sum_{cyc} \frac{x^3}{(x+2)^3} \geq \sum_{cyc} \frac{2x-1}{27} = \frac{4}{27}.$$

This ends the proof. Equality holds for $x=y=z=t=1$ or $a=b=c=d$.

∇

8.4 Symmetric separation

Review example 8.3.4 in the normalization section.

How does one final the following inequality, which solves the problem earl?

$$\frac{1}{2a^2-6a+9} \leq \frac{1}{5} - \frac{2(a-1)}{25} \;?$$

In this part, we will explain how to use "symmetric separation", an approach that has been used previously.

This approach can help us solve many problems which are represented in the form

$$f(x_1) + f(x_2) + \dots + f(x_n) \geq 0.$$

Indeed, to prove such inequalities, we will find functions $g(x)$ such that $f(x) \geq g(x)$ and

$$g(x_1) + g(x_2) + \dots + g(x_n) \geq 0.$$

Let's consider the following example

Example 8.4.1. *Let a, b, c be positive real numbers such that $abc = 1$. Prove that*

$$\frac{1}{3a^2+(a-1)^2}+\frac{1}{3b^2+(b-1)^2}+\frac{1}{3c^2+(c-1)^2} \geq 1.$$

(Le Huu Dien Khue)

SOLUTION. We want to find a real constant k such that

$$\frac{1}{3a^2+(a-1)^2} \geq \frac{1}{3}+k\ln a.$$

If there exists such a valid number k, we can conclude (notice $\ln a+\ln b+\ln c=0$)

$$\sum_{cyc}\frac{1}{3a^2+(a-1)^2} \geq 1+k\left(\sum_{cyc}\ln a\right)=1.$$

Denote

$$f(x)=\frac{1}{3x^2+(x-1)^2}-k\ln x-\frac{1}{3}.$$

Notice that $a=b=c=1$ makes up one case of equality. We predict that such a number k will bring about $f'(1)=0$. Since

$$f'(x)=\frac{-8x+2}{(3x^2+(x-1)^2)^2}-\frac{k}{x},$$

we infer $k=\dfrac{-2}{3}$. In this case,

$$f'(x)=\frac{-8x+2}{(3x^2+(x-1)^2)^2}+\frac{2}{3x}=\frac{2(x-1)(16x^3-1)}{3x\,(3x^2+(x-1)^2)^2}.$$

Unfortunately, the equation $f'(x)=0$ has more than one root, so the inequality $f(x)\geq 0$ is wrong (in fact, if we let $x\to 0$ then $f(x)\to -\infty$). However, from the derivative of $f(x)$, we can at least obtain that $f(x)\geq f(1)=0$ for all $x\in\left[\dfrac{1}{2},+\infty\right)$. So if all a,b,c belong to $\left[\dfrac{1}{2},+\infty\right)$, we have are

$$\sum_{cyc}\frac{1}{3a^2+(a-1)^2} \geq \sum_{cyc}\left(\frac{1}{3}-\frac{2}{3}\cdot\ln a\right)=1.$$

What if some numbers among a,b,c are smaller than $\dfrac{1}{2}$? Suppose that $a\leq\dfrac{1}{2}$ then $3a^2+(a-1)^2\leq 1$ and the problem becomes obvious

$$\sum_{cyc}\frac{1}{3a^2+(a-1)^2} \geq \frac{1}{3a^2+(a-1)^2} \geq 1.$$

We are done. Equality holds for $a = b = c = 1$.

∇

Making up the estimation

$$\frac{1}{3a^2 + (a-1)^2} \ge \frac{1}{3} + k \ln a$$

is a technique called "symmetric separation". If it's hard to prove $\sum\limits_{cyc} f(x_i) \ge 0$, we can separate this sum into smaller components $f(x_i) \ge g(x_i)$ then prove that $\sum\limits_{cyc} g(x_i) \ge 0$. Also, $g(x)$ should be guessed from the data given in the hypothesis: if the hypothesis is $x_1 x_2 ... x_n = 1$, we may predict $g(x) = k \ln x$ (k is a constant); if the hypothesis is $x_1 + x_2 + ... + x_n = n$, we may predict $g(x) = k(x-1)$; if the hypothesis is $x_1^2 + x_2^2 + ... + x_n^2 = n$, we may predict $g(x) = k(x^2 - 1)$, etc. Notice that these predictions must also depend on the case of equality (for example, above predictions of $g(x)$ are based on the case $x_1 = x_2 = ... = x_n = 1$).

How do we figure out a valid number k? Suppose that $f(x)$ has derivative and you predict that case $x_1 = x_2 = ... = x_n = 1$ makes up the equality then, is given by $f'(1) = 0$.

Although in some situations, the inequality $f(x) \ge g(x)$ doesn't hold for all x in the required range, it can hold for a large range of x and the work left may easy, as in example 8.4.1 when we examined case $a \le \frac{1}{2}$.

Example 8.4.2. *Let a, b, c be positive real numbers such that $abc = 1$. Prove that*

$$\frac{1}{a^2 - a + 1} + \frac{1}{b^2 - b + 1} + \frac{1}{c^2 - c + 1} \le 3.$$

(Vu Dinh Quy)

SOLUTION. First, we will prove that

$$f(x) = \frac{1}{x^2 - x + 1} + \ln x - 1 \ge 0 \;\forall x \in (0, 1.8].$$

Indeed, we have

$$f'(x) = \frac{-2x + 1}{(x^2 - x + 1)^2} + \frac{1}{x} = \frac{(x-1)(x^3 - x^2 - 1)}{(x^2 - x - 1)^2}.$$

The equation $x^3 = x^2 + 1$ has exactly one real root in $(0, 2]$, so it's easy to infer that

$$\max_{0 \le x \le 1.8} f(x) = \max\{f(1), f(1.8)\} = 0.$$

Therefore we are done if all a, b, c are smaller than 1.8. Otherwise, we suppose that $a \geq 1.8$. If $a \geq 2$ then we have

$$\sum_{cyc} \frac{1}{a^2 - a + 1} \leq \frac{1}{2^2 - 2 + 1} + \frac{1}{\left(b - \frac{1}{2}\right)^2 + \frac{3}{4}} + \frac{1}{\left(c - \frac{1}{2}\right)^2 + \frac{3}{4}} \leq \frac{1}{3} + \frac{4}{3} + \frac{4}{3} = 3.$$

So it's enough to consider the case $1.8 \leq a \leq 2$. WLOG, suppose that $b \geq c$. Because $a \leq 2$, we must have $b \geq \frac{1}{\sqrt{2}}$. If $a \geq 1.9$ then we can conclude by

$$\sum_{cyc} \frac{1}{a^2 - a + 1} \leq \frac{1}{1.9^2 - 1.9 + 1} + \frac{1}{\frac{1}{2} - \frac{1}{\sqrt{2}} + 1} + \frac{4}{3} < 3.$$

If $a \leq 1.9$ then $b \geq \frac{1}{\sqrt{1.9}}$ and we also conclude by

$$\sum_{cyc} \frac{1}{a^2 - a + 1} \leq \frac{1}{1.8^2 - 1.8 + 1} + \frac{1}{\frac{1}{1.9} - \frac{1}{\sqrt{1.9}} + 1} + \frac{4}{3} < 3.$$

The proof is finished. Equality holds for $a = b = c = 1$.

Comment. Taking into account example 2.1.9 we can solve this problem differently. Notice that

$$\sum_{cyc} \frac{a^4}{a^4 + a^2 + 1} = \sum_{cyc} \frac{1}{\left(\frac{1}{a^2}\right)^2 + \left(\frac{1}{a^2}\right) + 1} \geq 1. \qquad (\star)$$

So

$$\sum_{cyc} \frac{a^2 + 1}{a^2 - a + 1} = 3 + \sum_{cyc} \frac{a}{a^2 - a + 1} \leq 4 \Rightarrow \sum_{cyc} \frac{1}{a^2 - a + 1} + \sum_{cyc} \frac{1}{a^2 + a + 1} \leq 4.$$

Because by $(\star)$ $\sum_{cyc} \frac{1}{a^2 + a + 1} \geq 1$ we conclude $\sum_{cyc} \frac{1}{a^2 - a + 1} \leq 3.$

∇

Example 8.4.3. *Suppose that a, b, c, d, e, f are six positive real numbers satisfying $abcdef = 1$. Prove the following inequality*

$$\frac{2a + 1}{a^2 + a + 1} + \frac{2b + 1}{b^2 + b + 1} + \frac{2c + 1}{c^2 + c + 1} + \frac{2d + 1}{d^2 + d + 1} + \frac{2e + 1}{e^2 + e + 1} + \frac{2f + 1}{f^2 + f + 1} \leq 4.$$

(Pham Kim Hung)

SOLUTION. Consider the following function with $x > 0$

$$f(x) = \frac{1 + 2x}{1 + x + x^2} + \frac{\ln x}{3} - 1.$$

We have certainly

$$f'(x) = \frac{-2x^2-2x+1}{(1+x+x^2)^2} + \frac{1}{3x} = \frac{(x-1)(x^3-3x^2-6x-1)}{3x(1+x+x^2)^2}.$$

Notice that the equation $x^3 = 3x^2 + 6x + 1$ has exactly one real root x_0 in $(1, +\infty)$. Since $4 \le x_0 \le 12$ we have

$$\max_{0<x\le 12} f(x) = \max\{f(1), f(12)\} = 0.$$

If all a, b, c, d, e, f are smaller than 12 then we conclude that

$$\sum_{cyc} \frac{1+2a}{1+a+a^2} \le \sum_{cyc} \left(1 + \frac{\ln a}{3}\right) = 6.$$

Otherwise, suppose that $a \ge 12$. Notice that

$$7(1+x+x^2) - 6(1+2x) = 7x^2 - 5x + 1 > 0 \ \forall x \in \mathbb{R},$$

so we deduce

$$\sum_{cyc} \frac{1+2a}{1+a+a^2} \le \frac{1+2\cdot 12}{1+12+12^2} + \frac{5\cdot 7}{6} < 6.$$

We are done. Equality holds for $a = b = c = d = e = f = 1$.

∇

Example 8.4.4. *Suppose that a, b, c, d are positive real numbers satisfying $abcd = 1$. Prove that*

$$\frac{1+a}{1+a^2} + \frac{1+b}{1+b^2} + \frac{1+c}{1+c^2} + \frac{1+d}{1+d^2} \le 4.$$

(Vasile Cirtoaje)

SOLUTION. Consider the following function with $x > 0$

$$f(x) = \frac{1+x}{1+x^2} + \frac{\ln x}{2} - 1$$

We obtain

$$f'(x) = \frac{(x-1)(x^3-x^2-3x-1)}{2x(1+x^2)^2}.$$

Since the equation $x^3 = x^2 + 3x + 1$ has exactly one real root x_0 and $4 \ge x_0 > 1$ it's easy to get

$$\max_{0<x\le 4} f(x) = \max\{f(1), f(4)\} = 0.$$

If all a, b, c, d are smaller than 4 then we are done because

$$\sum_{cyc} \frac{1+a}{1+a^2} \le \sum_{cyc} \left(1 - \frac{\ln x}{2}\right) = 4.$$

Now suppose that $a \geq 4$. Since $21(1+x^2)-17(1+x) = 21x^2-17x+4 > 0$, it follows that

$$\sum_{cyc} \frac{1+a}{1+a^2} \geq \frac{1+4}{1+4^2} + \frac{3 \cdot 21}{17} = 4.$$

The proof is finished. Equality holds for $a=b=c=d=1$.

∇

Example 8.4.5. *Suppose that a,b,c,d,e,f are six positive real numbers satisfying $abcdef=1$. Prove the following inequality*

$$\frac{a-1}{a^2+a+1} + \frac{b-1}{b^2+b+1} + \frac{c-1}{c^2+c+1} + \frac{d-1}{d^2+d+1} + \frac{e-1}{e^2+e+1} + \frac{f-1}{f^2+f+1} \leq 0.$$

(Pham Kim Hung)

SOLUTION. Consider the following function with $x > 0$

$$f(x) = \frac{x-1}{x^2+x+1} - \frac{\ln x}{3}.$$

Notice that

$$f'(x) = \frac{(x-1)(-x^3-6x^2-3x+1)}{3x(x^2+x+1)^2}.$$

The equation $x^3+6x^2+3x=1$ has exactly one real root x_0 and clearly, $x_0 > \frac{1}{5}$. Therefore

$$\max_{1 \geq x \geq \frac{1}{11}} f(x) = \max\left\{f(1), f\left(\frac{1}{11}\right)\right\} = 0.$$

If all a,b,c,d,e,f are greater than $\frac{1}{11}$, we are done because

$$\sum_{cyc} \frac{a-1}{a^2+a+1} \leq \sum_{cyc} \frac{\ln a}{3} = 0.$$

Now suppose that $a \leq \frac{1}{11}$. Because the function $g(x) = \frac{x-1}{x^2+x+1}$ has the derivative $g'(x) = \frac{-x^2+2x+2}{(x^2+x+1)^2}$, we infer $g(a) \leq g\left(\frac{1}{11}\right)$ and

$$\max_{x>0} g(x) = g\left(1+\sqrt{3}\right)$$

and therefore we conclude that

$$g(a)+g(b)+g(c)+g(d)+g(e)+g(f) \leq g\left(\frac{1}{11}\right) + 5g\left(1+\sqrt{3}\right) < 0.$$

∇

Example 8.4.6. *Suppose that a, b, c, d, e, f, g are positive real numbers such that $a + b + c + d + e + f + g = 7$. Prove that*

$$(a^2 - a + 1)(b^2 - b + 1)(c^2 - c + 1)(d^2 - d + 1)(e^2 - e + 1)(f^2 - f + 1)(g^2 - g + 1) \geq 1.$$

(Pham Kim Hung)

SOLUTION. The inequality is equivalent to

$$\sum_{cyc} \ln(a^2 - a + 1) \geq 0.$$

Consider the function $f(x) = \ln(x^2 - x + 1) - x + 1$. We have

$$f'(x) = \frac{(x-1)(2-x)}{x^2 + x + 1}.$$

It's easy to get that

$$\min_{0 \leq x \leq 2.75} f(x) = \min\{f(1), f(2.75)\} = 0.$$

If all a, b, c, d, e, f, g are smaller than 2.75 then we are done

$$\sum_{cyc} \ln(a^2 - a + 1) \geq \sum_{cyc} (1 - a) = 0.$$

Otherwise, suppose that $a \geq 2.75$. Since $x^2 - x + 1 \geq \frac{3}{4}\ \forall x \in \mathbb{R}$,

$$\sum_{cyc} \ln(a^2 - a + 1) \geq \ln\left(2.75^2 - 2.75 + 1\right) + 6 \cdot \ln\left(\frac{3}{4}\right) > 0.$$

The proof is finished. Equality holds for $a = b = c = d = e = f = g = 1$.

∇

Example 8.4.7. *Let a, b, c, d be positive real numbers such that $a + b + c + d = 4$. Prove that*

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2} \geq a^2 + b^2 + c^2 + d^2.$$

(Pham Kim Hung)

SOLUTION. Consider the following function of positive variable x

$$f(x) = \frac{1}{x^2} - x^2 + 4x - 4.$$

We clearly have

$$f'(x) = \frac{-2}{x^3} - 2x + 4 = \frac{-2(x-1)(x^3 - x^2 - x - 1)}{x^3}.$$

The equation $f'(x) = 0$ has exactly two positive real roots. One root is 1 and one root is a number greater than 1.

$$\max_{0<x\le 2.4} f(x) = \max\{f(1)\ ;\ f(2.4)\} = 0.$$

If a, b, c, d are smaller than 2.4, we get the desired result since

$$\sum_{cyc} \frac{1}{a^2} - \sum_{cyc} a^2 = \sum_{cyc} f(a) \ge 0.$$

Otherwise, we may assume that $a \ge 2.4 \ge b \ge c \ge d$. Since $b + c + d \le 1.6$, we get

$$\frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2} \ge \frac{27}{(b+c+d)^2} > 10.$$

(i). The first case. $a \le 3$. We have the desired result since

$$a^2 + b^2 + c^2 + d^2 \le a^2 + (b+c+d)^2 =$$

$$= a^2 + (4-a)^2 \le 3^2 + 1^2 = 10,$$

since $3 \ge a \ge 1$.

(ii). The second case. If $a \ge 3$. Similarly, we get

$$\frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2} \ge \frac{27}{(b+c+d)^2} \ge 27 > 16 > a^2 + b^2 + c^2 + d^2.$$

Therefore the inequality is proved in every case. Equality holds for $a = b = c = d = 1$.

∇

Chapter 9

Problems and Solutions

After reading the previous 8 chapters, to discover a higher level of inequalities. In this chapter, there are 100 collected inequalities from recent mathematics contests and creations of some authors from all over the world. I hope they will depict a colorful picture so that you can appreciate their beauty. Perhaps, solving thoroughly 100 problems will cost you a remarkable amount of time, so, just see them as an interesting game of imagination, play with them rather than "work with them". If you can solve 70 problems, you are really good. If you can solve 90 problems, you are absolutely brilliant. If you can solve all, you must be a genius of inequalities. Take your time.

Problem 1. *Let a, b, c be non-negative real numbers such that $a + b + c = 3$. Prove the following inequality*

$$(a^2 - ab + b^2)(b^2 - bc + c^2)(c^2 - ca + a^2) \le 12.$$

(Pham Kim Hung)

SOLUTION. WLOG, assume that $a \ge b \ge c$. Certainly, we have

$$b^2 - bc + c^2 \le b^2,$$
$$a^2 - ac + c^2 \le c^2.$$

It suffices to prove that

$$M = a^2b^2(a^2 - ab + b^2) \le 12.$$

Denote $x = \dfrac{a-b}{2} \ge 0$ and $s = \dfrac{a+b}{2} \le \dfrac{3}{2}$. Rewrite M into the form

$$M = (s^2 - x^2)^2(s^2 + 3x^2).$$

Applying **AM-GM** inequality, we obtain

$$\frac{3}{2}(s^2 - x^2) \cdot \frac{3}{2}(s^2 - x^2) \cdot (s^2 + 3x^2) \le \left(\frac{4}{3}s^2\right)^3 \le 27 \Rightarrow \frac{9}{4}M \le 27 \Rightarrow M \le 12.$$

Equality holds for $a = 2, b = 1, c = 0$ up to permutation.

Comment. By the same approach, we can prove the following results

★ *Let a, b, c be non-negative real numbers. Prove that*

$$(a) \quad \frac{a^2}{b^2 - bc + c^2} + \frac{b^2}{c^2 - ca + a^2} + \frac{c^2}{a^2 - ab + b^2} \ge 2.$$

$$(b) \quad \frac{a}{\sqrt{b^2 - bc + c^2}} + \frac{b}{\sqrt{c^2 - ca + a^2}} + \frac{c}{\sqrt{a^2 - ab + b^2}} \ge 2.$$

$$(c) \quad \frac{1}{b^2 - bc + c^2} + \frac{1}{c^2 - ca + a^2} + \frac{1}{a^2 - ab + b^2} \ge \frac{6}{ab + bc + ca}.$$

★ *Let a, b, c be non-negative real numbers and $a^2 + b^2 + c^2 = 2$. Prove that*

$$8(a^2 - ab + b^2)(b^2 - bc + c^2)(c^2 - ca + a^2) \le 1.$$

To prove them, we carry out the same procedure. Suppose that $a \ge b \ge c$ then $a^2 - ac + c^2 \le a^2$ and $b^2 - bc + c^2 \le b^2$. The problems are changed to simpler forms in two variables a and b only; the remaining work is easy.

∇

Problem 2. *Suppose that a, b, c are positive real numbers. Prove that*

$$\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\left(\frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c}\right) \geq \frac{9}{1+abc}.$$

(Walther Janous)

Solution. By **Rearrangement** inequality,

$$\frac{1}{a(1+a)}+\frac{1}{b(1+b)}+\frac{1}{c(1+c)} \geq \frac{1}{b(1+c)}+\frac{1}{c(1+a)}+\frac{1}{a(1+b)},$$

Hence the inequality will be proved if these relations are fulfilled

$$\sum_{cyc} \frac{1}{b(1+c)} \geq \frac{3}{1+abc} \ ; \ \sum_{cyc} \frac{1}{a(1+c)} \geq \frac{3}{1+abc}.$$

We choose to prove the first inequality (the second one can be proved similarly). Let $a = \frac{ky}{x}, b = \frac{kz}{y}, c = \frac{kx}{z}$. Therefore the inequality is rewritten into the form

$$\sum_{cyc} \frac{1}{\frac{kz}{y}+\frac{k^2x}{y}} \geq \frac{3}{1+k^3} \Leftrightarrow \sum_{cyc} \frac{y}{z+kx} \geq \frac{3k}{1+k^3}.$$

According to **Cauchy-Schwarz** inequality, we have

$$\sum_{cyc} \frac{y}{z+kx} \geq \frac{(x+y+z)^2}{(k+1)(xy+yz+zx)} \geq \frac{3}{k+1}.$$

So it suffices to prove that

$$\frac{3}{k+1} \geq \frac{3k}{1+k^3} \Leftrightarrow (k-1)^2(k+1) \geq 0,$$

which is obvious. The equality holds for $x = y = z, k = 1 \Leftrightarrow a = b = c = 1$.

∇

Problem 3. *Let a, b, c be positive real numbers satisfying $a^2+b^2+c^2 = 3$. Prove that*

$$\frac{a}{a^2+2b+3}+\frac{b}{b^2+2c+3}+\frac{c}{c^2+2a+3} \leq \frac{1}{2}.$$

(Pham Kim Hung)

Solution. We certainly have $a^2+1 \geq 2a,\ b^2+1 \geq 2b,\ c^2+1 \geq 2c$, therefore

$$\sum_{cyc} \frac{a}{a^2+2b+3} \geq \sum_{cyc} \frac{a}{2(a+b+1)}.$$

It remains to prove that

$$\sum_{cyc}\frac{a}{a+b+1}\le 1 \Leftrightarrow \sum_{cyc}\frac{b+1}{a+b+1}\ge 2.$$

Notice that $a^2+b^2+c^2=3$, so

$$\sum_{cyc}(b+1)(a+b+1)=3+3\sum_{cyc}a+\sum_{cyc}ab+\sum_{cyc}a^2=\frac{1}{2}(a+b+c+3)^2.$$

According to **Cauchy-Schwarz** inequality, we deduce that

$$\sum_{cyc}\frac{b+1}{a+b+1}\ge\frac{(a+b+c+3)^2}{(b+1)(a+b+1)+(c+1)(b+c+1)+(a+1)(c+a+1)}=2.$$

The proof is finished. Equality holds for $a=b=c=1$.

∇

Problem 4. *Let $x_1,x_2,...,x_n$ be positive real numbers and $x_1x_2...x_n=1$. Prove that*

$$x_1+x_2+...+x_n\ge\frac{2}{1+x_1}+\frac{2}{1+x_2}+...+\frac{2}{1+x_n}.$$

(Pham Kim Hung)

Solution. Because $2-\dfrac{2}{1+x_i}=\dfrac{2x_i}{1+x_i}$, the problem can be rewritten to

$$\sum_{cyc}x_i+\sum_{cyc}\frac{2x_i}{x_i+1}\ge 2n.$$

According to **AM-GM** inequality, we conclude that

$$-2n+\sum_{cyc}x_i+\sum_{cyc}\frac{2x_i}{x_i+1}=-2n+\sum_{cyc}\left(\frac{x_i+1}{2}+\frac{2x_i}{1+x_i}\right)+\sum_{cyc}\frac{x_i-1}{2}\ge$$

$$\ge-\frac{5n}{2}+\sum_{cyc}\sqrt{x_i}+\frac{1}{2}\sum_{cyc}x_i\ge\frac{-3n}{2}+2n\sqrt[2n]{\prod_{cyc}x_i}+\frac{n}{2}\sqrt[n]{\prod_{cyc}x_i}=0.$$

This ends the proof. Equality holds for $x_1=x_2=...=x_n=1$.

∇

Problem 5. *Prove that for all positive real numbers $a,b,c\in[1,2]$, we have*

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\ge 6\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right).$$

(Tran Nam Dung, Viet Nam TST 2006)

SOLUTION. Instead of the condition $a, b, c \in [1, 2]$, we will prove this inequality with a stronger condition that a, b, c are the side lengths of a triangle. Using the identities

$$\left(\sum_{cyc} a\right)\left(\sum_{cyc} \frac{1}{a}\right) - 9 = \sum_{cyc} \frac{(a-b)^2}{ab} \quad , \quad 6\left(\sum_{cyc} \frac{a}{b+c}\right) - 3 = \sum_{cyc} \frac{(a-b)^2}{(a+c)(b+c)},$$

the inequality is equivalent to

$$S_a(b-c)^2 + S_b(c-a)^2 + S_c(a-b)^2 \geq 0$$

where S_a, S_b, S_c can be defined as

$$S_a = \frac{1}{bc} - \frac{3}{(a+b)(a+c)} \ ; \ S_b = \frac{1}{ca} - \frac{3}{(b+c)(b+a)} \ ; \ S_c = \frac{1}{ab} - \frac{3}{(c+a)(c+b)}.$$

WLOG, suppose that $a \geq b \geq c$, hence $S_a \geq S_b \geq S_c$. Notice that

$$S_b + S_c = \frac{1}{a}\left(\frac{1}{b} + \frac{1}{c}\right) - \frac{3}{b+c}\left(\frac{1}{a+b} + \frac{1}{a+c}\right) \geq 0$$

$$\Leftrightarrow \ \frac{1}{b} + \frac{1}{c} \geq \frac{3}{b+c}\left(\frac{a}{a+b} + \frac{a}{a+c}\right).$$

Because $a \leq b + c$, we have

$$\text{RHS} \leq \frac{3}{b+c}\left(\frac{b+c}{2b+c} + \frac{b+c}{2c+b}\right) = \frac{3}{2b+c} + \frac{3}{2c+b}.$$

By **Cauchy-Schwarz** inequality, we get

$$\frac{1}{b} + \frac{1}{c} - \left(\frac{3}{2b+c} + \frac{3}{2c+b}\right) = \frac{1}{3}\left(\frac{1}{b} + \frac{2}{c} - \frac{9}{2c+b}\right) + \frac{1}{3}\left(\frac{1}{c} + \frac{2}{b} - \frac{9}{2b+c}\right) \geq 0.$$

We conclude $S_b + S_c \geq 0$, which implies that $S_a \geq S_b \geq 0$ and

$$S_a(b-c)^2 + S_b(c-a)^2 + S_c(a-b)^2 \geq (S_b + S_c)(a-b)^2 \geq 0.$$

Equality holds for $a = b = c$ or $a = 2b = 2c$ up to permutation.

∇

Problem 6. *Let a, b, c be positive real numbers such that $a + b + c = 3$. Prove that*

$$a^2 + b^2 + c^2 \geq \frac{2+a}{2+b} + \frac{2+b}{2+c} + \frac{2+c}{2+a}.$$

(Pham Kim Hung)

SOLUTION. *(Cauchy reverse)* By **AM-GM**, we deduce that

$$\sum_{cyc}\frac{2+a}{2+b}=\sum_{cyc}\frac{2+a}{2}-\sum_{cyc}\frac{b(2+a)}{2(2+b)}\geq\frac{9}{2}-\frac{3\sqrt[3]{abc}}{2}.$$

So it's enough to prove that

$$\sum_{cyc}a^2+\frac{3\sqrt[3]{abc}}{2}\geq\frac{(a+b+c)^2}{2}$$

$$\Leftrightarrow\ \sum_{cyc}a^2+3(a+b+c)\sqrt[3]{abc}\geq 2\sum_{cyc}ab.$$

Let $x=\sqrt[3]{a}, y=\sqrt[3]{b}$ and $z=\sqrt[3]{c}$. By **AM-GM** inequality and the sixth **Schur** inequality, we have the desired result immediately

$$\sum_{cyc}x^6+xyz\sum_{cyc}x^3\geq\sum_{cyc}x^5(y+z)=\sum_{cyc}xy(x^4+y^4)\geq 2\sum_{cyc}x^3y^3.$$

The equality holds for $a=b=c=1$.

∇

Problem 7. *Let x,y,z be non-negative real numbers with sum 3. Prove that*

$$\sqrt{\frac{x}{1+2yz}}+\sqrt{\frac{y}{1+2zx}}+\sqrt{\frac{z}{1+2xy}}\geq\sqrt{3}.$$

(Phan Thanh Viet)

SOLUTION. According to **Cauchy-Schwarz** inequality, we deduce that

$$\sum_{cyc}\sqrt{\frac{x}{1+2yz}}=\sum_{cyc}\frac{x^2}{\sqrt{x}\sqrt{x^2+2x^2yz}}$$

$$\geq\frac{(x+y+z)^2}{\sqrt{x}\cdot\sqrt{x^2+2x^2yz}+\sqrt{y}\cdot\sqrt{y^2+2y^2zx}+\sqrt{z}\cdot\sqrt{z^2+2z^2xy}}$$

$$\geq\frac{(x+y+z)^2}{\sqrt{(x+y+z)\left(x^2+y^2+z^2+2xyz(x+y+z)\right)}}.$$

So it's enough to prove that

$$\left(\sum_{cyc}x\right)^3\geq 3\left(\sum_{cyc}x^2\right)+6xyz\left(\sum_{cyc}x\right)$$

$$\Leftrightarrow\ \left(\sum_{cyc}x\right)^3\geq\left(\sum_{cyc}x\right)\left(\sum_{cyc}x^2\right)+6xyz\left(\sum_{cyc}x\right)\ \Leftrightarrow\ 3\sum_{cyc}x(y-z)^2\geq 0,$$

which is obvious. The proof is finished and equality holds for $a=b=c=1$ and $a=3, b=c=0$ up to permutation.

∇

Problem 8. *Let $a_1, a_2, ..., a_n$ be positive real numbers and $a_1a_2...a_n = 1$. Prove that*

$$\sqrt{1+a_1^2}+\sqrt{1+a_2^2}+...+\sqrt{1+a_n^2} \le \sqrt{2}(a_1+a_2+...+a_n).$$

(Gabriel Dospinescu)

SOLUTION. From the obvious inequality $(\sqrt{x}-1)^4 \ge 0$, we see that

$$\frac{1+x^2}{2} \le (x-\sqrt{x}+1)^2 \Rightarrow \sqrt{\frac{1+x^2}{2}}+\sqrt{x} \le 1+x.$$

According to this result, we have of course

$$\sum_{i=1}^{n}\sqrt{1+a_i^2} \le \sqrt{2}\sum_{i=1}^{n}a_i + \sqrt{2}\left(n-\sum_{i=1}^{n}\sqrt{a_i}\right) \le \sqrt{2}\left(\sum_{i=1}^{n}a_i\right)$$

because **AM-GM** inequality yields that $\sum\limits_{i=1}^{n}\sqrt{a_i} \ge n$. The conclusion follows and equality holds if $a_1 = a_2 = ... = a_n = 1$.

Comment. We can solve this problem by using symmetric separation. Indeed, consider the following function

$$f(x) = \sqrt{1+x^2}-\sqrt{2}x+\left(\sqrt{2}-\frac{1}{\sqrt{2}}\right)\ln x.$$

Since

$$\begin{aligned} f'(x) &= \frac{x}{\sqrt{1+x^2}}-\sqrt{2}+\left(\sqrt{2}-\frac{1}{\sqrt{2}}\right)\cdot\frac{1}{x} \\ &= \frac{(x-1)\left(2x^3+x-1-2x^2\sqrt{2(1+x^2)}\right)}{x\left(\sqrt{2}x^2+\sqrt{1+x^2}\right)\sqrt{2(1+x^2)}}. \end{aligned}$$

Notice that $1+2x^2\sqrt{2(1+x^2)} \ge 1+2x^2(1+x) \ge 2x^3+x$ so we infer that $f'(x) = 0 \Leftrightarrow x = 1$. It's then easy to deduce that $\max\limits_{x>0} f(x) = f(1) = 0$. Therefore

$$\sum_{i=1}^{n}\sqrt{1+a_i^2} \le \sqrt{2}\sum_{i=1}^{n}a_i - \left(\sqrt{2}-\frac{1}{\sqrt{2}}\right)\sum_{i=1}^{n}\ln a_i = \sqrt{2}\sum_{i=1}^{n}a_i.$$

∇

Problem 9. *Let a, b, c, k be positive real numbers. Prove that*

$$\frac{a+kb}{a+kc}+\frac{b+kc}{b+ka}+\frac{c+ka}{c+kb} \le \frac{a}{b}+\frac{b}{c}+\frac{c}{a}.$$

(Nguyen Viet Anh)

SOLUTION. We denote

$$X = \frac{1+k\cdot\frac{a}{b}}{1+k},\ Y = \frac{1+k\cdot\frac{b}{c}}{1+k},\ Z = \frac{1+k\cdot\frac{c}{a}}{1+k}.$$

According to **Hölder** inequality, we get

$$\prod_{cyc}\left(1+\frac{ka}{b}\right) \geq (1+k)^3,$$

or equivalently $XYZ \geq 1$. Now rewrite the inequality into the following form

$$\sum_{cyc}\left(\frac{a}{b}-\frac{c+ka}{c+kb}\right) \geq 0 \Leftrightarrow \sum_{cyc}\frac{c(a-b)}{b(c+kb)} \geq 0 \Leftrightarrow \sum_{cyc}\frac{\frac{a}{b}-1}{1+\frac{kb}{c}} \geq 0$$

$$\Leftrightarrow \sum_{cyc}\frac{\left(\frac{1}{k}+1\right)(X-1)}{(k+1)Y} \geq 0 \Leftrightarrow \frac{X}{Y}+\frac{Y}{Z}+\frac{Z}{X} \geq \frac{1}{X}+\frac{1}{Y}+\frac{1}{Z}$$

which is true according to **AM-GM** inequality because

$$3\sum_{cyc}\frac{X}{Y} = \sum_{cyc}\left(\frac{X}{Y}+\frac{X}{Y}+\frac{Z}{X}\right) \geq 3\sum_{cyc}\sqrt[3]{\frac{XZ}{Y^2}} = 3\sum_{cyc}\frac{1}{Y}.$$

Equality holds for $X = Y = Z = 1$ or equivalent to $a = b = c$.

∇

Problem 10. *Let a, b, c be positive real numbers such that $abc = 1$. Prove that*

$$\frac{1}{(a+1)(a+2)}+\frac{1}{(b+1)(b+2)}+\frac{1}{(c+1)(c+2)} \geq \frac{1}{2}.$$

(Pham Kim Hung)

SOLUTION. By hypothesis $abc = 1$, so there are three positive real numbers x, y, z for which $a = \frac{yz}{x^2}, b = \frac{zx}{y^2}, c = \frac{xy}{z^2}$. The inequality becomes

$$\sum_{cyc}\frac{x^4}{(x^2+yz)(2x^2+yz)} \geq \frac{1}{2}.$$

According to **Cauchy-Schwarz** inequality, we deduce that

$$\text{LHS} \geq \frac{(x^2+y^2+z^2)^2}{(x^2+yz)(2x^2+yz)+(y^2+zx)(2y^2+zx)+(z^2+xy)(2z^2+xy)}.$$

It remains to prove that

$$2(x^2+y^2+z^2)^2 \geq \sum_{cyc}(x^2+yz)(2x^2+yz)$$

$$\Leftrightarrow 3\sum_{cyc} x^2y^2 \geq 3\sum_{cyc} x^2yz \Leftrightarrow \sum_{cyc} x^2(y-z)^2 \geq 0,$$

which is obvious. Equality holds for $x = y = z$ or $a = b = c = 1$.

∇

Problem 11. *Let a, b, c, d be non-negative real numbers such that $a + b + c + d = 4$. Prove that*

$$a^2 + b^2 + c^2 + d^2 - 4 \geq 4(a-1)(b-1)(c-1)(d-1).$$

(Pham Kim Hung)

SOLUTION. Applying **AM-GM** inequality, we get

$$\begin{aligned} a^2 + b^2 + c^2 + d^2 - 4 &= (a-1)^2 + (b-1)^2 + (c-1)^2 + (d-1)^2 \\ &\geq 4\sqrt{|a-1)(b-1)(c-1)(d-1)|}. \end{aligned}$$

It suffices to consider the inequality in the case $a \geq b \geq 1 \geq c \geq d$ (in order to have $(a-1)(b-1)(c-1)(d-1) \geq 0$). Since $a + b \leq 4$ and $c, d \leq 1$,

$$(1-c)(1-d) \leq 1 ;$$

$$(a-1)(b-1) \leq \frac{1}{4}(a+b-2)^2 \leq 1 ;$$

Therefore we reach the desired result since

$$\sqrt{|(a-1)(b-1)(c-1)(d-1)|} \geq (a-1)(b-1)(c-1)(d-1)$$

Equality holds for $a = b = c = d = 1$, $a = b = 2, c = d = 0$ and any permutations.

∇

Problem 12. *Let x, y, z be distinct real numbers. Prove that*

$$\frac{x^2}{(x-y)^2} + \frac{y^2}{(y-z)^2} + \frac{z^2}{(z-x)^2} \geq 1.$$

(Le Huu Dien Khue)

SOLUTION. We have

$$\sum_{cyc}\left(1-\frac{x}{z}\right)^2\left(1-\frac{z}{y}\right)^2 - \left(1-\frac{y}{x}\right)^2\left(1-\frac{z}{y}\right)^2\left(1-\frac{x}{z}\right)^2 =$$

$$= \sum_{cyc}\left(1-\frac{x}{z}-\frac{z}{y}+\frac{x}{y}\right)^2 - \left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}-\frac{y}{x}-\frac{z}{y}-\frac{x}{z}\right)^2 =$$

$$=3+2\sum_{cyc}\frac{x^2}{z^2}+\sum_{cyc}\frac{x^2}{y^2}-4\sum_{cyc}\frac{x}{z}+4\sum_{cyc}\frac{x}{y}-2\sum_{cyc}\frac{x^2}{yz}-2\sum_{cyc}2\frac{yz}{x^2}$$

$$-\sum_{cyc}\frac{x^2}{y^2}-\sum_{cyc}\frac{y^2}{x^2}+2\left(\sum_{cyc}\frac{x}{y}\right)\left(\sum_{cyc}\frac{y}{x}\right)-2\sum_{cyc}\frac{y}{x}-2\sum_{cyc}\frac{x}{y}=$$

$$=\sum_{cyc}\frac{x^2}{z^2}-6\sum_{cyc}\frac{x}{z}+2\sum_{cyc}\frac{x}{y}+9$$

$$=\left(\frac{x}{z}+\frac{z}{y}+\frac{y}{x}-3\right)^2\geq 0.$$

We conclude that

$$\sum_{cyc}\left(1-\frac{x}{z}\right)^2\left(1-\frac{z}{y}\right)^2\geq\left(1-\frac{y}{x}\right)^2\left(1-\frac{z}{y}\right)^2\left(1-\frac{x}{z}\right)^2\Rightarrow$$

$$\Rightarrow\sum_{cyc}x^2(z-x)^2(z-y)^2\geq(x-y)^2(y-z)^2(z-x)^2\Rightarrow$$

$$\Rightarrow\frac{x^2}{(x-y)^2}+\frac{y^2}{(y-z)^2}+\frac{z^2}{(z-x)^2}\geq 1,$$

as desired. Equality holds for all triples (x,y,z) such that $\frac{x}{z}+\frac{z}{y}+\frac{y}{x}=3$.

∇

Problem 13. *Let a,b,c,d be non-negative real numbers with sum 3. Prove that*

$$ab(b+c)+bc(c+d)+cd(d+a)+da(a+b)\leq 4.$$

(Pham Kim Hung)

Solution. WLOG, we may assume that $b+d\leq a+c$. We have

$$ab(b+c)+bc(c+d)+cd(d+a)+da(a+b)=$$
$$=(a+c)(bc+da)+(b+d)(ab+cd)=$$
$$=(a+c)\left((a+c)(b+d)-(ab+cd)\right)+(b+d)(ab+cd)=$$
$$=(a+c)^2(b+d)+(ab+cd)(b+d-a-c)\leq(a+c)^2(b+d).$$

Finally, AM-GM inequality shows that

$$2(a+c)^2(b+d)=(a+c)(a+c)(2b+2d)\leq 8\ \Rightarrow(a+c)(b+d)^2\leq 4.$$

Equality holds for $a=1,b=2,c=d=0$ and permutations.

∇

Problem 14. *Let $a_1, a_2, ..., a_n$ be arbitrary real numbers. Prove that*

$$\sum_{i,j=1}^{n} |a_i + a_j| \geq n \sum_{i=1}^{n} |a_i|.$$

(Iran TST 2006)

SOLUTION. Separating the sequence $(a_1, a_2, ..., a_n)$ into two sub-sequences of non-negative and elements

$$\{a_1, a_2, ..., a_n\} = \{b_1, b_2, ..., b_r\} \cup \{-c_1, -c_2, ..., -c_s\},$$

with $n = r + s,\ b_i \geq 0\ \forall i \in \{1, 2, ..., r\},\ c_i > 0\ \forall j \in \{1, 2, ..., s\}$.

Let $R = \sum\limits_{i=1}^{r} b_i$ and $S = \sum\limits_{j=1}^{s} c_j$. The inequality becomes

$$2\sum_{i=1}^{r}\sum_{j=1}^{s} |b_i - c_j| + 2r\sum_{i=1}^{r} b_i + 2s\sum_{j=1}^{s} c_j \geq n\left(\sum_{i=1}^{r} b_i + \sum_{j=1}^{s} c_j\right)$$

$$\Leftrightarrow\ 2\sum_{i=1}^{r}\sum_{j=1}^{s} |b_i - c_j| \geq (s - r)(R - S).$$

WLOG, suppose that $s \geq r$. Clearly, we only need to consider the case $R \geq S$. Hence

$$\sum_{i=1}^{r}\sum_{j=1}^{s} |b_i - c_j| \geq \sum_{i=1}^{r} (sb_i - S) = sR - rS.$$

We will prove that

$$2(sR - rS) \geq (s - r)(R - S)\ \Leftrightarrow\ S(s - r) + r(R - S) + sR - rS \geq 0,$$

which is obviously true because $s \geq r$ and $R \geq S$. Equality holds if and only if $|a_1| = |a_2| = ... = |a_n|$ and in the set $\{a_1, a_2, ..., a_n\}$, the number of the negative terms is equal to the number of the non-negative terms.

∇

Problem 15. *Let a, b, c be non-negative real numbers. Prove that*

$$3(a + b + c) \geq 2\left(\sqrt{a^2 + bc} + \sqrt{b^2 + ca} + \sqrt{c^2 + ab}\right).$$

(Pham Kim Hung)

SOLUTION. WLOG, we may assume that $a \geq b \geq c$. Hence

$$\sqrt{b^2+ca}+\sqrt{c^2+ab} \leq \sqrt{2(b^2+c^2)+2a(b+c)}.$$

We need to prove that

$$2\sqrt{2(b^2+c^2)+2a(b+c)}+2\sqrt{a^2+bc} \leq 3(a+b+c).$$

Let $s=\frac{1}{2}(b+c)$. Squaring both hand sides, we obtain an equivalent inequality

$$8(b^2+c^2+2as) \leq 9(a+2s)^2+2(a^2+bc)-12(a+2s)\sqrt{a^2+bc}$$

$$\Leftrightarrow\ (a-2s)^2+20bc \geq 12(a+2s)\left(\sqrt{a^2+bc}-a\right).$$

Clearly,

$$\sqrt{a^2+bc}-a=\frac{bc}{a+\sqrt{a^2+bc}} \leq \frac{bc}{2a}.$$

So it suffices to prove that

$$(a-2s)^2+20bc \geq \frac{6(a+2s)bc}{a}$$

$$\Leftrightarrow\ (a-2s)^2+2bc+\frac{12(a-s)bc}{a} \geq 0.$$

which is obvious because $a \geq s$. Equality holds for $a=b, c=0$ or permutations.

∇

Problem 16. *Suppose that a, b, c are three non-negative real numbers. Prove that*

$$\frac{1}{a^2+b^2}+\frac{1}{b^2+c^2}+\frac{1}{c^2+a^2} \geq \frac{10}{(a+b+c)^2}.$$

(Vasile Cirtoaje, Nguyen Viet Anh)

SOLUTION. WLOG, assume that $c=\min(a,b,c)$, then

$$b^2+c^2 \leq \left(b+\frac{c}{2}\right)^2=x^2,$$
$$a^2+c^2 \leq \left(a+\frac{c}{2}\right)^2=y^2,$$
$$a^2+b^2 \leq \left(a+\frac{c}{2}\right)^2+\left(b+\frac{c}{2}\right)^2=x^2+y^2.$$

We deduce that

$$\text{LHS} \geq \left(\frac{1}{x^2}+\frac{1}{y^2}\right)\cdot\frac{3}{4}+\left(\frac{1}{x^2}+\frac{1}{y^2}\right)\cdot\frac{1}{4}+\frac{1}{x^2+y^2} \geq$$
$$\geq \frac{\frac{3}{4}\cdot 8}{(x+y)^2}+\frac{1}{2xy}+\frac{1}{x^2+y^2} \geq \frac{6}{(x+y)^2}+\frac{(x+y)^2}{2(x^4+y^4)} \geq$$
$$\geq \frac{10}{(x+y)^2},$$

We used Hölder: $(x+y)(x+y)\left(\frac{1}{x^2}+\frac{1}{y^2}\right)\geq 8$. Equality holds for $a=b, c=0$ or permutations.

Comment. This solution can help us create a more general problem

★ *Let $a_1, a_2, ..., a_n$ be non-negative real numbers. Find the maximum k such that the following inequality holds*

$$\frac{1}{a_2^2+a_3^2+...+a_n^2}+\frac{1}{a_1^2+a_3^2+...+a_n^2}+\dots+\frac{1}{a_1^2+a_2^2+...+a_{n-1}^2}\geq\frac{k}{(a_1+a_2+...+a_n)^2}.$$

WLOG, suppose that $a_1\geq a_2\geq ...\geq a_n$. Denote $a=a_1+\frac{1}{2}\sum_{i=3}^{n}a_i$ and $b=a_2+\frac{1}{2}\sum_{i=3}^{n}a_i$. Clearly, $a_1^2+\sum_{j=3}^{n}a_j^2\leq a^2; b^2+\sum_{j=3}^{n}a_j^2\leq b^2$ and for all $k\in\{3,4,...,n\}$

$$a_1^2+a_2^2+\sum_{j=1,j\neq k}^{n}a_j^2\leq a^2+b^2.$$

Case $a_3=a_4=...=a_n=0$ makes up the equality in all above inequalities. Therefore it's sufficient to examine the following expression for positive real numbers a, b with $a+b=1$

$$A=\frac{n-2}{a^2+b^2}+\frac{1}{a^2}+\frac{1}{b^2}.$$

Denote $x=a^2+b^2$ $(x\geq\frac{1}{2})$ then $2ab=1-x$, therefore

$$A=\frac{n-2}{x}+\frac{4x}{(1-x)^2}=f(x)$$

Notice that $x\geq\frac{1}{2}$, so

$$f'(x)=\frac{-n+2}{x^2}+\frac{4(1+x)}{(1-x)^3}\geq 0,\ \forall n\leq 14.$$

Hence if $n\in\{3,4,5,...,14\}$, we can conclude that

$$f(x)\geq f\left(\frac{1}{2}\right)=2n+4\ \Rightarrow k=2n+4.$$

If $n\geq 15$, the function $4x^2(1+x)-(n-2)(1-x)^3=0$ has exactly one real roots greater than $\frac{1}{2}$ (because $f\left(\frac{1}{2}\right)<0$). Suppose that x_0 is this root, then k is exactly

$$k=\min_{\frac{1}{2}\leq x<1}f(x)=f(x_0)=\frac{n-2}{x_0}+\frac{4x_0}{(1-x_0)^2}$$

We can prove by a similar method the next problem.

★ *Let a, b, c, d be positive real numbers. Prove that*

$$\frac{1}{a^3+b^3}+\frac{1}{a^3+c^3}+\frac{1}{a^3+d^3}+\frac{1}{b^3+c^3}+\frac{1}{b^3+d^3}+\frac{1}{c^3+d^3} \geq \frac{243}{2(a+b+c+d)^3}.$$

∇

Problem 17. *Let a, b, c, d be positive real numbers such that $abcd = 1$. Prove that*

$$(1+a^2)(1+b^2)(1+c^2)(1+d^2) \geq (a+b+c+d)^2.$$

(Pham Kim Hung)

SOLUTION. Because $abcd = 1$, there are two numbers between a, b, c, d both not smaller than 1 or not bigger than 1. WLOG, suppose that they are b and d, then $(b-1)(d-1) \geq 0$. Applying **Cauchy-Schwarz** inequality, we get

$$\begin{aligned}(1+a^2)(1+b^2)(1+c^2)(1+d^2) &= (1+a^2+b^2+a^2b^2)(c^2+1+d^2+c^2d^2) \\ &\geq (c+a+bd+1)^2 \geq (a+b+c+d)^2.\end{aligned}$$

Equality holds for $a = b = c = d = 1$.

∇

Problem 18. *Let a, b, c be positive real numbers such that $a + b + c = 1$. Prove that*

$$\frac{ab}{\sqrt{ab+bc}}+\frac{bc}{\sqrt{bc+ca}}+\frac{ca}{\sqrt{ca+ab}} \leq \frac{1}{\sqrt{2}}.$$

(Chinese MO 2006)

SOLUTION. The above inequality is equivalent to

$$\sum_{cyc} a \cdot \sqrt{\frac{b}{a+c}} \leq \frac{1}{\sqrt{2}} \Leftrightarrow \sum_{cyc} \frac{a+b}{2} \cdot \sqrt{\frac{a^2b}{(a+c)(a+b)^2}} \leq \frac{1}{\sqrt{2}}.$$

Using weighted **Jensen** inequality for the concave function $f(x) = \sqrt{x}$, we obtain

$$\sum_{cyc} \frac{a+b}{2} \cdot \sqrt{\frac{4a^2b}{(a+c)(a+b)^2}} \leq \sqrt{\sum_{cyc} \frac{2a^2b}{(a+b)(a+c)}}.$$

It remains to prove that

$$\sum_{cyc} \frac{a^2b}{(a+b)(a+c)} \leq \frac{1}{4}$$

$$\Leftrightarrow 4\sum_{cyc} a^2b(b+c) \leq (a+b+c)\prod_{cyc}(a+b)$$

$$\Leftrightarrow\ 2\sum_{cyc} a^2b^2 \le \sum_{cyc} a^3(b+c),$$

which is obvious. Equality holds for $a=b=c=\dfrac{1}{3}$.

∇

Problem 19. *Let x,y,z be non-negative real numbers such that $x+y+z=1$. Find the maximum of*

$$\frac{x-y}{\sqrt{x+y}}+\frac{y-z}{\sqrt{y+z}}+\frac{z-x}{\sqrt{z+x}}$$

(Pham Kim Hung)

SOLUTION. First we consider the problem in the case $\min(x,y,z)=0$. WLOG, suppose that $z=0$, then $x+y=1$ and therefore

$$\sum_{cyc}\frac{x-y}{\sqrt{x+y}}=\frac{x-y}{\sqrt{x+y}}+\sqrt{y}-\sqrt{x}=x-y+\sqrt{y}-\sqrt{x}=u(v-1),$$

in which $u=\sqrt{x}-\sqrt{y}, v=\sqrt{x}+\sqrt{y}$ and $u^2+v^2=2$.

Denote $u^2(v-1)^2=(2-v^2)(v-1)^2=f(v)$ We have $f'(v)=2(v-1)(2+v-2v^2)$ and it's easy to infer that

$$\max_{1\le v\le\sqrt{2}} = f\left(\frac{1+\sqrt{17}}{4}\right)=\frac{71-17\sqrt{17}}{32}.$$

So if $\min(x,y,z)=0$ then the maximum we are looking for is

$$k=\sqrt{\frac{71-17\sqrt{17}}{32}}.$$

This result also shows that if $\min(x,y,z)=0$ then

$$\frac{x-y}{\sqrt{x+y}}+\frac{y-z}{\sqrt{y+z}}+\frac{z-x}{\sqrt{z+x}}\le k\sqrt{x+y+z}\ (\star)$$

Now we will prove $(\star)$ for all non-negative real numbers x,y,z (we dismiss the condition $x+y+z=1$ because the inequality is homogeneous). Denote $c=\sqrt{x+y},\ b=\sqrt{x+z},\ a=\sqrt{y+z}$. The problem can be rewritten in the form

$$\frac{b^2-a^2}{c}+\frac{c^2-b^2}{a}+\frac{a^2-c^2}{b}\le\frac{k}{\sqrt{2}}\cdot\sqrt{a^2+b^2+c^2}\Leftrightarrow$$

$$\Leftrightarrow\ (a-b)(b-c)(c-a)\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right)\le\frac{k}{\sqrt{2}}\cdot\sqrt{a^2+b^2+c^2}\ (\star\star)$$

WLOG, suppose that $c = \max(a, b, c)$. If $a \geq b$, then the conclusion follows immediately. Otherwise, suppose that $b \geq a$. Because $c^2 \leq b^2 + a^2$, there exists an unique real number $t \leq a$ for which $(a-t), (b-t), (c-t)$ are side lengths of a right triangle (that means $(a-t)^2 + (b-t)^2 = (c-t)^2$). Clearly, if we replace a, b, c with $a-t, b-t, c-t$, the left-hand expression of $(\star\star)$ is increased but the right-hand expression of $(\star\star)$ is decreased, so we conclude that it is enough to consider $(\star\star)$ in the case when a, b, c are three side lengths of a right triangle. That means $a^2 + b^2 = c^2$ or $z = 0$. But the case $z = 0$ has been proved above, so we are done.

∇

Problem 20. *Let x, y, z be three real numbers in $[-1, 1]$ such that $x + y + z = 0$. Prove that*

$$\sqrt{1 + x + y^2} + \sqrt{1 + y + z^2} + \sqrt{1 + z + x^2} \geq 3.$$

(Phan Thanh Nam)

SOLUTION. First, we will prove that if $ab \geq 0$ then

$$\sqrt{1+a} + \sqrt{1+b} \geq 1 + \sqrt{1+a+b}.$$

Indeed, after squaring, the inequality becomes

$$2 + a + b + 2\sqrt{(1+a)(1+b)} \geq 2 + a + b + 2\sqrt{1+a+b}$$

$$\Leftrightarrow\ (1+a)(1+b) \geq 1 + a + b\ \Leftrightarrow\ ab \geq 0,$$

and we are done. Notice that between $x + y^2, y + z^2, z + x^2$, at least two numbers have the same sign. WLOG, we may assume that $(x + y^2)(y + z^2) \geq 0$, then the above result shows that:

$$\begin{aligned}
&\sqrt{1 + x + y^2} + \sqrt{1 + y + z^2} + \sqrt{1 + z + x^2} \\
&\qquad \geq 1 + \sqrt{1 + x + y^2 + y + z^2} + \sqrt{1 + z + x^2} \\
&\qquad = 1 + \sqrt{\left(\sqrt{1 - z + z^2}\right)^2 + y^2} + \sqrt{\left(\sqrt{1+z}\right)^2 + x^2} \\
&\qquad \geq 1 + \sqrt{\left(\sqrt{1 - z + z^2} + \sqrt{1+z}\right)^2 + (x+y)^2} \\
&\qquad = 1 + \sqrt{\left(\sqrt{1 - z + z^2} + \sqrt{1+z}\right)^2 + z^2}.
\end{aligned}$$

It remains to prove that

$$\left(\sqrt{1 - z + z^2} + \sqrt{1+z}\right)^2 + z^2 \geq 4$$

$$\Leftrightarrow\ 2z^2+2\sqrt{1+z^3}\ge 2\ \Leftrightarrow\ z^2(2-z)(z+1)\ge 0,$$

which is clearly true because $|z|\le 1$. Equality holds for $x=y=z=0$.

Comment. By a similar approach, we can prove the same result with four numbers

★ *Let x,y,z,t be real numbers in $[-1,1]$ such that $x+y+z+t=0$. Prove that*

$$\sqrt{1+x+y^2}+\sqrt{1+y+z^2}+\sqrt{1+z+t^2}+\sqrt{1+t+x^2}\ge 4.$$

∇

Problem 21. *Suppose that a,b,c are three non-negative real numbers satisfying $ab+bc+ca=1$. Prove the inequality*

$$\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\ge\frac{5}{2}.$$

(Berkeley Mathematics Circle)

Solution. We denote $x=a+b+c$ and $z=abc$. The inequality becomes

$$2\sum_{cyc}(a+b)(a+c)\ge 5\prod_{cyc}(a+b)\ \Leftrightarrow\ 6+2\sum_{cyc}a^2\ge 5(a+b+c-abc)$$

$$\Leftrightarrow\ 2x^2-5x+2+5z\ge 0\ \Leftrightarrow\ (x-2)(2x-1)+5z\ge 0.$$

If $x\ge 2$, we are done. Otherwise, suppose that $x\le 2$. Because

$$(a+b-c)(b+c-a)(c+a-b)=(x-2a)(x-2b)(x-2c)\le abc.$$

we obtain $9z\ge 4x-x^3$. So it's enough to prove that

$$(x-2)(2x-1)+5\frac{4x-x^3}{9}\ge 0\Leftrightarrow(x-2)[18x-9-5x(2+x)]\ge 0\Leftrightarrow$$

$$\Leftrightarrow(x-2)[-5x^2+8x-9]\ge 0$$

which holds because $x\le 2$. Equality holds for $a=b=1,c=0$ or permutations.

Comment. We have a nice and similar result as follows

★ *Let a,b,c be non-negative real numbers and $ab+bc+ca=1$. Prove that*

$$\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b+c}\ge 3.$$

Proof. If $a+b+c\le 2$ then this problem deduced from the above result. Now suppose that $a+b+c\ge 2$ and $a\ge b\ge c$, then

$$\frac{1}{a+b+c}+\sum_{cyc}\frac{1}{a+b}=\frac{1}{a+b}+\frac{ab+bc+ca}{b+c}+\frac{ab+bc+ca}{c+a}+\frac{1}{a+b+c}=$$

$$= \frac{1}{a+b} + a + b + \frac{c(1+ab)}{1+c^2} + \frac{1}{a+b+c} \ge \frac{1}{a+b} + \frac{1}{a+b+c} + (a+b+c) \ge$$

$$\ge \left(\frac{1}{a+b} + \frac{a+b}{4}\right) + \left(\frac{1}{a+b+c} + \frac{a+b+c}{4}\right) + \left(\frac{a+b+c}{2}\right) \ge 1+1+1 = 3.$$

This ends the proof. Equality holds for $a = b = 1, c = 0$ up to permutation.

∇

Problem 22. *Prove the following inequality*

$$(\sqrt{2})^n (a_1 + a_2)(a_2 + a_3)...(a_n + a_1) \le (a_1 + a_2 + a_3)(a_2 + a_3 + a_4)....(a_n + a_1 + a_2),$$

where $a_1, a_2, ..., a_n$ are arbitrary positive real numbers.

(Russia MO)

SOLUTION. According to the following results

$$(a_1 + a_2 + a_3)^2 \ge (2a_1 + a_2)(a_2 + 2a_3),$$
$$(2a_1 + a_2)(2a_2 + a_1) = 2a_1^2 + 2a_2^2 + 5a_1a_2 \ge 2(a_1 + a_2)^2,$$

we are done immediately because

$$2^n \prod_{cyc} (a_1 + a_2)^2 \le \prod_{cyc} (2a_1 + a_2)(2a_2 + a_1)$$

$$= \prod_{cyc} (2a_1 + a_2)(a_2 + 2a_3) \le \prod_{cyc} (a_1 + a_2 + a_3)^2.$$

∇

Problem 23. *Let a, b, c be non-negative real numbers with sum 3. Prove that*

$$\frac{ab+bc+ca}{a^3b^3 + b^3c^3 + c^3a^3} \ge \frac{a^3 + b^3 + c^3}{36}.$$

(Pham Kim Hung, MYM)

SOLUTION. WLOG, we may assume that $a \ge b \ge c$. Denote

$$f(a,b,c) = 36(ab+bc+ca) - (a^3 + b^3 + c^3)(a^3b^3 + b^3c^3 + c^3a^3).$$

We will prove $f(a,b,c) \ge f(a, b+c, 0)$. Indeed,

$$a^3 + b^3 + c^3 \le a^3 + (b+c)^3 \quad ; \quad a^3b^3 + b^3c^3 + c^3a^3 \le a^3(b+c)^3$$

$$\Rightarrow (a^3 + b^3 + c^3)(a^3b^3 + b^3c^3 + c^3a^3) \le \left(a^3 + (b+c)^3\right) a^3(b+c)^3.$$

Also $ab + bc + ca \geq a(b+c)$. We obtain $f(a,b,c) \geq f(a, b+c, 0)$. It remains to prove the first inequality in the case $c = 0$, or namely

$$36ab \geq a^3b^3(a^3+b^3) \Leftrightarrow 36 \geq a^2b^2(a^3+b^3).$$

Let $x = ab$.The inequality can be rewritten in the form

$$t^2(27-9t) \leq 36 \Leftrightarrow \frac{t}{2} \cdot \frac{t}{2}(3-t) \leq 1,$$

which is exactly **AM-GM** inequality. Equality holds for $c = 0$ and $a + b = 3, ab = 2$, or equivalently $a = 2, b = 1, c = 0$ (and its permutations of course).

∇

Problem 24. *Let a, b, c, d be four real numbers satisfying that $(1+a^2)(1+b^2)(1+c^2)(1+d^2) = 16$. Prove the inequality*

$$-3 \leq ab + bc + ca + da + ac + bd - abcd \leq 5.$$

(Titu Andreescu and Gabriel Dospinescu)

SOLUTION. We denote $S = ab + bc + cd + da + ac + bd - abcd$, then

$$S - 1 = (1-ab)(cd-1) + (a+b)(c+d).$$

Applying **Cauchy-Schwarz** inequality, we obtain

$$(S-1)^2 \leq \left((1-ab)^2 + (a+b)^2\right)\left((1-cd)^2 + (c+d)^2\right)$$
$$= (1+a^2)(1+b^2)(1+c^2)(1+d^2) = 16.$$

Hence $|S-1| \leq 4$ or equivalently $-3 \leq S \leq 5$. Equality can happen; for example, $(a,b,c,d) = (1,-1,1,-1)$ and $(1,1,1,1)$.

∇

Problem 25. *Let a, b, c, d be positive real numbers such that $abcd = 1$. Prove that*

$$\frac{1}{(1+a)^2} + \frac{1}{(1+b)^2} + \frac{1}{(1+c)^2} + \frac{1}{(1+d)^2} \geq 1.$$

(China TST 2004)

SOLUTION. First, notice that for any non-negative real numbers x, y

$$\frac{1}{(1+x)^2} + \frac{1}{(1+y)^2} \geq \frac{1}{1+xy}.$$

By expanding, the above inequality becomes

$$(2+2x+2y+x^2+y^2)(1+xy) \geq (1+2x+x^2)(1+2y+y^2)$$

$$\Leftrightarrow \; xy(x^2+y^2)+1 \geq 2xy+x^2y^2$$

$$\Leftrightarrow \; (xy-1)^2+xy(x-y)^2 \geq 0.$$

Let $m = ab, n = cd \Rightarrow mn = 1$, therefore

$$\frac{1}{1+m}+\frac{1}{1+n} = \frac{m+n+2}{(m+1)(n+1)} = 1.$$

Using these two results, we conclude that

$$\frac{1}{(1+a)^2}+\frac{1}{(1+b)^2}+\frac{1}{(1+c)^2}+\frac{1}{(1+d)^2} \geq \frac{1}{1+m}+\frac{1}{1+n} = 1.$$

Equality holds for $a = b = c = d = 1$.

Comment. 1. We can also solve this problem by **Cauchy-Schwarz**. Indeed, there exist four positive real numbers s, t, u, v such that

$$a = \frac{stu}{v^3}, b = \frac{tuv}{s^3}, c = \frac{uvs}{t^3}, a = \frac{vst}{u^3},$$

and the problem can be rewritten as

$$\sum_{cyc} \frac{v^6}{(v^3+stu)^2} \geq 1.$$

According to
textcolorculCauchy-Schwartz, it's enough to prove that

$$\left(\sum_{cyc} v^3\right)^2 \geq \sum_{cyc}(v^3+stu)^2$$

$$\Leftrightarrow \; \sum_{cyc} v^3(s^3+t^3+u^3) \geq 2\sum_{cyc} v^3stu + \sum_{cyc} s^2t^2u^2.$$

This last inequality is obtained from the following results

$$\sum_{cyc} v^3(s^3+t^3+u^3) \geq 3\sum_{cyc} v^3stu \quad ;$$

$$\sum_{cyc} v^3(s^3+t^3+u^3) = \sum(s^3t^3+t^3u^3+u^3s^3) \geq 3\sum_{cyc} s^2t^2u^2 \quad ;$$

and the proof is finished. Equality holds for $s = t = u = v$ or $a = b = c = d = 1$.

2. The previous solution also helps us create a similar problem as follow

★ *Let $a_1, a_2, ..., a_9$ be positive real numbers with product* 1. *Prove that*

$$\frac{1}{(2a_1+1)^2}+\frac{1}{(2a_2+1)^2}+...+\frac{1}{(2a_9+1)^2}\ge 1.$$

Moreover, the general result is also valid (and solved through the same method)

★ *Let $a_1, a_2, ..., a_n$ be positive real numbers with product* 1. *For $k=\sqrt{n}-1$, prove that*

$$\frac{1}{(ka_1+1)^2}+\frac{1}{(ka_2+1)^2}+...+\frac{1}{(ka_n+1)^2}\ge 1.$$

∇

Problem 26. *Let a, b, c be positive real numbers. Prove that*

$$\frac{a+2b}{c+2b}+\frac{b+2c}{a+2c}+\frac{c+2a}{b+2a}\ge 3.$$

(Pham Kim Hung)

SOLUTION. By expanding, we can rewrite the inequality to

$$\sum_{cyc}(a+2b)(a+2c)(b+2a)\ge 3\prod_{cyc}(c+2b)$$

$$\Leftrightarrow\ 2(a^3+b^3+c^3)+3abc\ge 3(a^2b+b^2c+c^2a),$$

which is a combination of the third degree-Schur inequality and **AM-GM** inequality

$$2\sum_{cyc}a^3+3abc-3\sum_{cyc}a^2b=3abc+\sum_{cyc}a^3-\sum_{cyc}a^2(b+c)+\sum_{cyc}(a^3+ab^2-2a^2b)\ge 0.$$

This ends the proof. Equality holds for $a=b=c$.

∇

Problem 27. *Let a, b, c be three positive real numbers. Prove that*

$$\frac{a^4}{a^2+ab+b^2}+\frac{b^4}{b^2+bc+c^2}+\frac{c^4}{c^2+ca+a^2}\ge\frac{a^3+b^3+c^3}{a+b+c}.$$

(Phan Thanh Viet)

SOLUTION. Notice that

$$\frac{a^3+b^3+c^3}{a+b+c}=\frac{3abc}{a+b+c}+a^2+b^2+c^2-ab-bc-ca.$$

Therefore, the inequality can be rewritten in the following form

$$\sum_{cyc}\left(\frac{a^4}{a^2+ab+b^2}-a^2+ab\right)\geq\frac{3abc}{a+b+c}\Leftrightarrow\sum_{cyc}\frac{ab^3}{a^2+ab+b^2}\geq\frac{3abc}{a+b+c}.$$

According to **Cauchy-Schwarz**, we deduce that

$$\sum_{cyc}\frac{ab^3}{a^2+ab+b^2}=\sum_{cyc}\frac{b^2}{1+\frac{a}{b}+\frac{b}{a}}\geq$$

$$\geq\frac{(a+b+c)^2}{3+\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{b}{a}+\frac{c}{b}+\frac{a}{c}}=\frac{abc(a+b+c)}{ab+bc+ca}.$$

It remains to prove that

$$\frac{abc(a+b+c)}{ab+bc+ca}\geq\frac{3abc}{a+b+c}\Leftrightarrow(a+b+c)^2\geq 3(ab+bc+ca)$$

which is obviously true. Equality holds for $a=b=c$.

∇

Problem 28. *Prove that for all non-negative real numbers a,b,c*

$$a^2+b^2+c^2+2abc+1\geq 2(ab+bc+ca).$$

(Darij Grinberg)

SOLUTION. I will give four solutions to the above inequality.

First Solution. Transform the inequality into a quadratic form in a

$$f(a)=a^2+2(bc-b-c)a+(b-c)^2+1.$$

(i). If $bc-b-c\geq 0$, we get the desired result immediately.

(ii). If $bc-b-c\leq 0$ then $(b-1)(c-1)\leq 1$. Notice that

$$\Delta'_f=(bc-b-c)^2-(b-c)^2-1=bc(b-2)(c-2)-1.$$

If both b and c are smaller than 2 then by **AM-GM** inequality, we get

$$b(2-b)\leq 1\ ,\ c(2-c)\leq 1\quad\Rightarrow\quad\Delta'_f\leq 0.$$

Otherwise, suppose $b\geq 2$, then $c\leq 2$ and clearly, $\Delta'_f\leq 0$. We are done.

Second Solution. We denote $k=a+b+c$. According to the inequality

$$abc\geq(a+b-c)(b+c-a)(c+a-b)=(k-2a)(k-2b)(k-2c)$$

we obtain

$$4(ab+bc+ca)-k^2 \le \frac{9abc}{k} \ (\star)$$

The inequality is equivalent to

$$(a+b+c)^2+2abc+1 \ge 4(ab+bc+ca) \ \Leftrightarrow\ 4(ab+bc+ca)-k^2 \le 1+2abc.$$

Taking into account $(\star)$, it remains to prove that

$$\left(\frac{9}{k}-2\right)abc \le 1.$$

If $9 \le 2k$, we are done immediately. Otherwise, **AM-GM** inequality shows that

$$\left(\frac{9}{k}-2\right)abc \le \left(\frac{9}{k}-2\right)\frac{k^3}{27} = \frac{(9-2k)\cdot k\cdot k}{27} \le 1,$$

which is exactly the desired result. Equality holds for $a=b=c=1$.

Third Solution. Because $2abc+1 \ge 3\sqrt[3]{a^2b^2c^2}$, it remains to prove that

$$x^6+y^6+z^6+3x^2y^2z^2 \ge 2(x^3y^3+y^3z^3+z^3x^3)$$

where $a=x^3,\ b=y^3,\ c=z^3$. According to **Schur** inequality, we obtain

$$3x^2y^2z^2+\sum_{cyc}x^6 \ge \sum_{cyc}x^4(y^2+z^2)=\sum_{cyc}x^2y^2(x^2+y^2) \ge 2\sum_{cyc}x^3y^3.$$

Fourth Solution. Rewrite our inequality in the form

$$(a-1)^2+(b-1)^2+(c-1)^2 \ge 2(1-a)(1-b)(1-c).$$

If a,b,c are all greater than 1, the conclusion follows immediately. Otherwise, suppose $c \le 1$. **AM-GM** inequality shows

$$(a-1)^2+(b-1)^2 \ge 2|(a-1)(b-1)| \ge 2|(a-1)(b-1)|(1-c) \ge 2(a-1)(b-1)(c-1)$$

and we are done of course.

Comment. The following is a similar inequality

$$a^2+b^2+c^2+2abc+3 \ge (1+a)(1+b)(1+c).$$

∇

Problem 29. *Let a,b,c be positive real numbers and $a^2+b^2+c^2=3$. Prove that*

$$\frac{1}{2-a}+\frac{1}{2-b}+\frac{1}{2-c} \ge 3.$$

(Pham Kim Hung)

SOLUTION.

First Solution. *(We use Cauchy-Schwarz inequality)* Rewrite the inequality to

$$\left(\frac{1}{2-a}-\frac{1}{2}\right)+\left(\frac{1}{2-b}-\frac{1}{2}\right)+\left(\frac{1}{2-c}-\frac{1}{2}\right)\geq\frac{3}{2}$$

$$\Leftrightarrow\ \frac{a}{2-a}+\frac{b}{2-b}+\frac{c}{2-c}\geq 3$$

$$\Leftrightarrow\ \frac{a^4}{2a^3-a^4}+\frac{b^4}{2b^3-b^4}+\frac{c^4}{2c^3-c^4}\geq 3.$$

Applying **Cauchy-Schwarz** inequality, we conclude that

$$\text{LHS}\geq\frac{(a^2+b^2+c^2)^2}{2(a^3+b^3+c^3)-(a^4+b^4+c^4)}=\frac{9}{2(a^3+b^3+c^3)-(a^4+b^4+c^4)}$$

and we are done because $2\sum_{cyc}a^3-\sum_{cyc}a^4\leq\sum_{cyc}a^2=3$.

Second Solution. *(We use Cauchy reverse)* Notice that $a(2-a)\leq 1$, so

$$\frac{1}{2-a}=\frac{1}{2}+\frac{a}{2(2-a)}=\frac{1}{2}+\frac{a^2}{2a(2-a)}\geq\frac{1}{2}+\frac{a^2}{2}$$

therefore

$$\sum_{cyc}\frac{1}{2-a}\geq\frac{3}{2}+\frac{1}{2}\sum_{cyc}a^2=3.$$

We are done. Equality holds for $a=b=c=1$.

∇

Problem 30. *Let a,b,c,d be non-negative real numbers. Prove that*

$$\left(1+\frac{2a}{b+c}\right)\left(1+\frac{2b}{c+d}\right)\left(1+\frac{2c}{d+a}\right)\left(1+\frac{2d}{a+b}\right)\geq 9.$$

(Vasile Cirtoaje)

SOLUTION. Rewrite it into another form

$$\left(1+\frac{a+c}{a+b}\right)\left(1+\frac{a+c}{c+d}\right)\left(1+\frac{b+d}{b+c}\right)\left(1+\frac{b+d}{a+d}\right)\geq 9.$$

For all positive real numbers x,y, it's easy to see that

$$\left(1+\frac{1}{x}\right)\left(1+\frac{1}{y}\right)\geq\left(1+\frac{2}{x+y}\right)^2.$$

Thus we have

$$\left(1+\frac{a+c}{a+b}\right)\left(1+\frac{a+c}{c+d}\right)\geq\left(1+\frac{2(a+c)}{a+b+c+d}\right)^2,$$
$$\left(1+\frac{b+d}{b+c}\right)\left(1+\frac{b+d}{a+d}\right)\geq\left(1+\frac{2(b+d)}{a+b+c+d}\right)^2.$$

It remains to prove that

$$\left(1+\frac{2(a+c)}{a+b+c+d}\right)\left(1+\frac{2(b+d)}{a+b+c+d}\right)\geq 3,$$

which is obvious. Equality holds for $a=c=0, b=d$ or $a=c, b=d=0$.

Comment. Here is the general result

★ *Let a,b,c,d,k be non-negative real numbers. Prove that*

$$\left(1+\frac{ka}{b+c}\right)\left(1+\frac{kb}{c+d}\right)\left(1+\frac{kc}{d+a}\right)\left(1+\frac{kd}{a+b}\right)\geq(k+1)^2.$$

This inequality can be obtained from the results below

$$\sum_{cyc}\frac{a}{b+c}\geq 2\ (\star)\quad;\qquad\sum_{cyc}\frac{ab}{(b+c)(c+d)}\geq 1\ (\star\star)\quad;$$

Notice that $(\star)$ is **Nesbitt** inequality for four numbers which has been proved in the previous chapter. To prove $(\star\star)$, we note that it is equivalent to (after expanding)

$$\sum_{cyc}a^2b^2+\sum_{cyc}a^3b+\sum_{cyc}abc^2\geq\sum_{cyc}a^2bc.$$

This inequality is true because

$$\sum_{cyc}a^3b+\sum_{cyc}abc^2\geq 2\sum_{cyc}a^2bc.$$

We have equality if $a=c, b=d=0$ or $a=c=0, b=d$.

▽

Problem 31. *Let a,b,c be non-negative real numbers. Prove that*

$$\frac{1}{a^2+b^2}+\frac{1}{b^2+c^2}+\frac{1}{c^2+a^2}+\frac{8}{a^2+b^2+c^2}\geq\frac{6}{ab+bc+ca}.$$

(Pham Kim Hung)

SOLUTION. WLOG, assume that $a \ge b \ge c$. Denote $t=\sqrt{b^2+c^2}$ and

$$f(a,b,c)=\frac{1}{a^2+b^2}+\frac{1}{b^2+c^2}+\frac{1}{c^2+a^2}+\frac{8}{a^2+b^2+c^2}-\frac{6}{ab+bc+ca}.$$

We have

$$\begin{aligned}
f(a,b,c)-f(a,t,0) &= \frac{c^2}{(a^2+b^2)(a^2+t^2)}-\frac{c^2}{a^2(a^2+c^2)}+\frac{6}{a\sqrt{b^2+c^2}}-\frac{6}{ab+bc+ca}\\
&\ge \frac{6a\left(b+c-\sqrt{b^2+c^2}\right)}{a\sqrt{b^2+c^2}(ab+bc+ca)}-\frac{c^2}{a^2(a^2+c^2)}\\
&\ge \frac{6bc}{(b+c)\sqrt{b^2+c^2}(ab+bc+ca)}-\frac{c^2}{a^2(a^2+c^2)}\\
&\ge \frac{6bc}{\sqrt{2}(b^2+c^2)(ab+bc+ca)}-\frac{bc}{a^2(a^2+c^2)}\ge 0
\end{aligned}$$

because

$$3\sqrt{2}a^2(a^2+c^2)\ge (ab+bc+ca)(b^2+c^2).$$

According to **AM-GM inequality**, we have

$$f(a,t,0)=\frac{9}{a^2+t^2}+\frac{1}{a^2}+\frac{1}{t^2}-\frac{6}{at}=\frac{9}{a^2+t^2}+\frac{a^2+t^2}{a^2t^2}-\frac{6}{at}\ge 0.$$

Therefore, we are done. Equality holds for $(a,b,c)\sim\left(\frac{-3\pm\sqrt{5}}{2},1,0\right)$.

∇

Problem 32. *Let a,b,c,k be positive real numbers and $k\ge\frac{3}{2}$. Prove that*

$$\frac{a^k}{a+b}+\frac{b^k}{b+c}+\frac{c^k}{c+a}\ge\frac{1}{2}\left(a^{k-1}+b^{k-1}+c^{k-1}\right).$$

(Vasile Cirtoaje and Pham Kim Hung)

SOLUTION. I have two solutions to this problem.

First Solution. *(Cauchy reverse)* Rewrite the inequality in the form

$$\sum_{cyc}\left(a^{k-1}-\frac{a^k}{a+b}\right)\le\frac{1}{2}\left(\sum_{cyc}a^{k-1}\right)\Leftrightarrow\sum_{cyc}\frac{a^{k-1}b}{a+b}\le\frac{1}{2}\left(\sum_{cyc}a^{k-1}\right).$$

Notice that

$$\sum_{cyc}\frac{a^{k-1}b}{a+b}\le\sum_{cyc}\frac{a^{k-1}b}{2\sqrt{ab}}=\frac{1}{2}\left(\sum_{cyc}a^{k-\frac{3}{2}}b^{\frac{1}{2}}\right).$$

It suffices to prove that

$$\sum_{cyc} a^{k-\frac{3}{2}} b^{\frac{1}{2}} \le \sum_{cyc} a^{k-1}.$$

According to **AM-GM** inequality, for $2k-2$ variables and using $k \ge \frac{3}{2}$, we obtain

$$(2k-2)\sum_{cyc} a^{k-1} = \sum_{cyc}\left((2k-3)a^{k-1} + b^{k-1}\right) \ge \sum_{cyc}(2k-2)a^{k-\frac{3}{2}} b^{\frac{1}{2}}.$$

Second Solution. The inequality can be rewritten in the form

$$\frac{a^{k-1}(a-b)}{a+b} + \frac{b^{k-1}(b-c)}{b+c} + \frac{c^{k-1}(c-a)}{c+a} \ge 0.$$

Notice that for all positive real numbers a, b

$$\frac{a^{k-1}(a-b)}{a+b} \ge \frac{a^{k-1} - b^{k-1}}{2(k-1)} \quad (\star)$$

Indeed, this relation can be obtained directly by **AM-GM**

$$(2k-3)a^k + b^k + ab^{k-1} \ge (2k-1)a^{k-1}b.$$

According to $(\star)$, we conclude that

$$\sum_{cyc} \frac{a^{k-1}(a-b)}{a+b} \ge \sum_{cyc} \frac{a^{k-1} - b^{k-1}}{2(k-1)} = 0.$$

This ends the proof. Equality holds for $a = b = c$.

$$\nabla$$

Problem 33. *Let a, b, c be non-negative real numbers and $a+b+c=3$. Prove that*

$$a\sqrt{1+b^3} + b\sqrt{1+c^3} + c\sqrt{1+a^3} \le 5.$$

(Pham Kim Hung)

SOLUTION. By **AM-GM** inequality, we deduce that

$$\sum_{cyc} a\sqrt{1+b^3} = \sum_{cyc} a\sqrt{(1+b)(1-b+b^2)} \le \frac{1}{2}\sum_{cyc} a(1+b^2).$$

It remains to prove that

$$ab^2 + bc^2 + ca^2 \le 4.$$

WLOG, we may suppose that b is the middle number between a, b, c. That means $a(b-a)(b-c) \le 0$, or $ab^2 + a^2c \le abc + a^2b$. It then suffices to prove that

$$abc + a^2b + bc^2 \le 4 \;\Leftrightarrow\; b(a^2 + ac + c^2) \le 4.$$

According to **AM-GM** inequality, we have

$$b(a^2+ac+c^2) \le b(a+c)^2 = 4b\cdot\frac{(a+c)}{2}\cdot\frac{(a+c)}{2} \le 4\left(\frac{a+b+c}{3}\right)^3 = 4.$$

We are done. Equality holds for $a=1, b=2, c=0$ and its permutations.

∇

Problem 34. *Let a,b,c be positive real numbers such that $a+b+c=1$. Prove that*

$$\frac{a}{\sqrt{\frac{1}{b}-1}}+\frac{b}{\sqrt{\frac{1}{c}-1}}+\frac{c}{\sqrt{\frac{1}{a}-1}} \le \frac{3\sqrt{3}}{4}\cdot\sqrt{(1-a)(1-b)(1-c)}.$$

(Do Hoang Giang)

SOLUTION. Our inequality is equivalent to

$$P=\sum_{cyc}\sqrt{\frac{a}{(a+c)(a+b)}\cdot\frac{ab}{(b+c)(c+a)}} \le \frac{3\sqrt{3}}{4}.$$

WLOG, assume that $a=\min\{a,b,c\}$. Consider the following cases

(i). The first case. If $a \le b \le c$, simultaneously have

$$\frac{ab}{(b+c)(c+a)} \le \frac{ca}{(a+b)(b+c)} \le \frac{bc}{(c+a)(a+b)};$$

$$\frac{a}{(a+c)(a+b)} \le \frac{b}{(a+b)(b+c)} \le \frac{c}{(b+c)(c+a)};$$

So, according to **Rearrangement** inequality, we infer that

$$P \le \sqrt{\frac{a^2b}{(a+b)(a+c)^2(b+c)}}+\sqrt{\frac{abc}{(a+b)^2(b+c)^2}}+\sqrt{\frac{bc^2}{(c+a)^2(a+b)(b+c)}}$$

$$=\sqrt{\frac{abc}{(a+b)^2(b+c)^2}}+\frac{b}{(a+b)(b+c)}\left(\frac{a}{a+c}+\frac{c}{c+a}\right)$$

$$\le\sqrt{3\left(\frac{abc}{(a+b)^2(b+c)^2}+2\cdot\frac{1}{4}\cdot\frac{b}{(a+b)(b+c)}\right)}$$

(since $\sqrt{x}+\sqrt{y}+\sqrt{z} \le \sqrt{3(x+y+z)}$.

It remains to prove that

$$\frac{abc}{(a+b)^2(b+c)^2}+\frac{1}{2}\cdot\frac{b}{(a+b)(b+c)} \ge \frac{9}{16}$$

which can be transformed to

$$(3ac-b)^2 \ge 0.$$

(ii). The second case. If $a \le c \le b$, we have

$$\frac{ca}{(a+b)(b+c)} \le \frac{ab}{(b+c)(c+a)} \le \frac{bc}{(c+a)(a+b)}\,;$$

$$\frac{a}{(a+c)(a+b)} \le \frac{c}{(b+c)(c+a)} \le \frac{b}{(a+b)(b+c)}\,.$$

So, according to **Rearrangement** inequality, we infer that

$$P \le \sqrt{\frac{a^2c}{(a+c)(a+b)^2(b+c)}} + \sqrt{\frac{abc}{(a+c)^2(b+c)^2}} + \sqrt{\frac{b^2c}{(a+c)(a+b)^2(b+c)}}$$
$$\le \sqrt{3\left(\frac{abc}{(a+c)^2(b+c)^2} + 2\cdot\frac{1}{4}\cdot\frac{c}{(b+c)(c+a)}\right)}.$$

It remains to prove that

$$\frac{abc}{(a+c)^2(b+c)^2} + \frac{1}{2}\cdot\frac{c}{(b+c)(c+a)} \le \frac{9}{16}$$

which can be transformed to an obviousness

$$(3ab-c)^2 \ge 0.$$

The inequality is proved in every case. The equality holds for $a = b = c = \dfrac{1}{3}$.

$$\nabla$$

Problem 35. *Suppose that* a, b, c *are three non-negative real numbers verifying* $a^2 + b^2 + c^2 = 1$. *Prove the following inequality*

$$\frac{a}{a^3+bc} + \frac{b}{b^3+ac} + \frac{c}{c^3+ab} \ge 3.$$

SOLUTION. I have two solutions to this problem.

First Solution. If all terms in the left hand side are greater than 1 then the inequality is proved immediately. Otherwise, we may assume that

$$x = \frac{a}{a^3+bc} \le 1 \;\Rightarrow a \le a^3 + bc.$$

Applying **Cauchy-Schwarz** inequality, we obtain

$$\frac{b}{b^3+ac} + \frac{c}{c^3+ab} \ge \frac{4}{b^2+c^2+\frac{ac}{b}+\frac{ab}{c}} = \frac{4}{1+y},$$

where $y = \dfrac{ac}{b} + \dfrac{ab}{c} - a^2$. Notice that the relation $x \ge y$ is equivalent to

$$(a^3+bc)(b^2+c^2-abc) \le bc$$

$$\Leftrightarrow a^3(1-a^2-abc) \le bc(a^2+abc)$$

$$\Leftrightarrow a^2(1-a^2) \le bc(a^3+a+bc).$$

which is obvious because $a(1-a^2) \le bc$. We conclude that

$$\frac{a}{a^3+bc}+\frac{b}{b^3+ac}+\frac{c}{c^3+ab} \ge x+\frac{4}{1+x} = (x+1)+\frac{4}{x+1}-1 \ge 3.$$

The proof is finished. Equality holds for $a=1, b=c=0$ or permutations.

Second Solution. Denote $x=\frac{bc}{a}, y=\frac{ac}{b}, z=\frac{ab}{c}$, so we have $xy+yz+zx = a^2+b^2+c^2=1$. We have to prove that

$$\frac{1}{x+yz}+\frac{1}{y+xz}+\frac{1}{z+xy} \ge 3.$$

Denote $s=x+y+z$ and $p=xyz$. Notice that

$$\sum_{cyc}\frac{1}{x+yz} \ge \frac{9}{x+y+z+xy+yz+zx} = \frac{9}{s+1}.$$

If $s \le 2$. we are done. Now consider the case $s \ge 2$. After expanding and reducing similar terms, the inequality becomes

$$s+7sp \ge 2+3p^2+3ps^2 \Leftrightarrow (s-2)(1-3sp)+p(s-3p) \ge 0$$

which is clearly true because $s \ge 2, 1 \ge 3sp$ and $s \ge 3p$. The conclusion follows.

∇

Problem 36. *Let a, b, c be non-negative real numbers. Prove that*

$$\frac{b+c}{\sqrt{a^2+bc}}+\frac{c+a}{\sqrt{b^2+ca}}+\frac{a+b}{\sqrt{c^2+ab}} \ge 4.$$

(Pham Kim Hung)

SOLUTION. Applying **Hölder** inequality, we obtain

$$\left(\sum_{cyc}\frac{b+c}{\sqrt{a^2+bc}}\right)^2\left(\sum_{cyc}(b+c)(a^2+bc)\right) \ge 8\left(\sum_{cyc}a\right)^3.$$

Therefore, it's enough to prove that

$$(a+b+c)^3 \ge 4\sum_{cyc}a^2(b+c) \Leftrightarrow 6abc+\sum_{cyc}a^3 \ge \sum_{cyc}a^2(b+c),$$

which comes from the third-degree **Schur** inequality because

$$3abc+\sum_{cyc}a^3 \ge \sum_{cyc}a^2(b+c).$$

Equality holds for $a = b, c = 0$ up to permutations.

Comment. By the same method, we can prove the following result

★ *Let a, b, c be non-negative real numbers and $a + b + c = 2$. Prove that*

$$\frac{a}{\sqrt{3+b^2+c^2}} + \frac{b}{\sqrt{3+c^2+a^2}} + \frac{c}{\sqrt{3+a^2+b^2}} \geq 1.$$

∇

Problem 37. *Let a, b, c be non-negative real numbers satisfying $a^2+b^2+c^2+abc = 4$. Prove that $2 + abc \geq ab + bc + ca \geq abc$.*

(USA MO 2001)

SOLUTION. To prove the right hand inequality, just notice that at least one of a, b, c, say a, is not bigger than 1. Thereforec we have $ab+bc+ca \geq bc \geq abc$. Equality holds for $(a, b, c) = (0, 0, 2)$ up to permutation.

To prove the right hand inequality, notice that two numbers among a, b, c, say a and c, are not smaller than 1 or not bigger than 1. Therefore $b(a-1)(c-1) \geq 0 \Leftrightarrow abc + b \geq ab + bc$ and it suffices to prove $2 \geq ac + b$.

From the hypothesis, we have

$$a^2 + c^2 + b(ac+b) = 4 \Rightarrow 2ac + b(ac+b) \leq 4 \Rightarrow (b+2)(ac+b-2) \leq 0,$$

thus $ac + b \leq 2$ and the desired result follows. Equality holds for $a = b = c = 1$.

∇

Problem 38. *Let a, b, c be non-negative real numbers. Prove that*

$$\sqrt{\frac{a^2+2bc}{b^2+c^2}} + \sqrt{\frac{b^2+2ca}{c^2+a^2}} + \sqrt{\frac{c^2+2ab}{a^2+b^2}} \geq 3.$$

(Vo Quoc Ba Can, Vu Dinh Quy)

SOLUTION. WLOG, assume that $a \geq b \geq c$. First, we will prove that

$$\sqrt{\frac{a^2+c^2}{b^2+c^2}} + \sqrt{\frac{b^2+c^2}{c^2+a^2}} \geq \sqrt{\frac{a}{b}} + \sqrt{\frac{b}{a}}.$$

Indeed, this condition is equivalent to

$$\frac{a^2+c^2}{b^2+c^2} + \frac{b^2+c^2}{c^2+a^2} \geq \frac{a}{b} + \frac{b}{a}$$

$$\Leftrightarrow \quad \frac{(a-b)^2(a+b)(ab-c^2)}{ab(a^2+c^2)(b^2+c^2)} \ge 0,$$

which is true because $a \ge b \ge c$. Using this result, we have

$$\sqrt{\frac{a^2+2bc}{b^2+c^2}} + \sqrt{\frac{b^2+2ca}{c^2+a^2}} + \sqrt{\frac{c^2+2ab}{a^2+b^2}}$$
$$\ge \sqrt{\frac{a^2+c^2}{b^2+c^2}} + \sqrt{\frac{b^2+c^2}{c^2+a^2}} + \sqrt{\frac{2ab}{a^2+b^2}}$$
$$\ge \sqrt{\frac{a}{b}} + \sqrt{\frac{b}{a}} + \sqrt{\frac{2ab}{a^2+b^2}}.$$

Denote $x = \sqrt{\frac{a}{b}} + \sqrt{\frac{b}{a}} \ge 2$. If $x \ge 3$, we are done. Otherwise, assume that $x \le 3$. In this case, we need

$$\sqrt{\frac{a}{b}} + \sqrt{\frac{b}{a}} + \sqrt{\frac{2ab}{a^2+b^2}} = x + \sqrt{\frac{2}{x^2-2}} \ge 3$$

since

$$\frac{2}{x^2-2} - (3-x)^2 = \frac{(x-2)^2(-x^2+2x+5)}{x^2-3} \ge 0.$$

The inequality is proved, with equality for $a = b, c = 0$ or permutations.

∇

Problem 39. *Let a, b, c be three distinct positive real numbers. Prove that*

$$\frac{1}{|a^2-b^2|} + \frac{1}{|b^2-c^2|} + \frac{1}{|c^2-a^2|} + \frac{8}{a^2+b^2+c^2} \ge \frac{28}{(a+b+c)^2}.$$

(Pham Kim Hung)

SOLUTION. WLOG, we may assume that $a > b > c$. Notice that

$$\frac{1}{a^2-b^2} + \frac{1}{b^2-c^2} + \frac{1}{a^2-c^2} + \frac{8}{a^2+b^2+c^2} - \left(\frac{1}{a^2-b^2} + \frac{1}{b^2} + \frac{1}{a^2} + \frac{8}{a^2+b^2}\right)$$
$$= c^2\left(\frac{1}{a^2(a^2-c^2)} + \frac{1}{b^2(b^2-c^2)} - \frac{8}{(a^2+b^2+c^2)(a^2+b^2)}\right)$$
$$\ge c^2\left(\frac{1}{a^4} + \frac{1}{b^4} - \frac{8}{(a^2+b^2)^2}\right) \ge 0.$$

The right hand expression is a decreasing function of c, so it's enough to prove the inequality for $c = 0$

$$\frac{1}{a^2-b^2} + \frac{1}{b^2} + \frac{1}{a^2} + \frac{8}{a^2+b^2} \ge \frac{28}{(a+b)^2}$$

$$\Leftrightarrow\ 2\left(\frac{a}{b}+\frac{b}{a}\right)+\frac{a^2}{b^2}+\frac{b^2}{a^2}+\frac{16ab}{a^2+b^2}+\frac{a+b}{a-b}\ge 18.$$

Since the inequality is homogeneous, we can assume $a > b = 1$ and so the inequality becomes:

$$2\left(a+\frac{1}{a}\right)+a^2+\frac{1}{a^2}+\frac{16a}{a^2+1}+\frac{a+1}{a-1}\ge 18$$

$$\Leftrightarrow\ \frac{2(a-1)^2}{a}+\frac{(a^2-1)^2}{a^2}-\frac{8(a-1)^2}{a^2+1}+\frac{a+1}{a-1}\ge 4$$

$$\Leftrightarrow\ \frac{2(a-1)^4}{a(a^2+1)}+\frac{(a-1)^4(a+1)^2}{a^2(a^2+1)}+2(a-1)^2+\frac{a+1}{a-1}\ge 4.$$

If $a \le \frac{5}{3}$ then $\frac{a+1}{a-1} \ge 4$ and the desired result follows immediately. Otherwise, we have $a \ge \frac{4}{3}$. According to **AM-GM** inequality, we get

$$2(a-1)^2+\frac{a+1}{a-1}=2(a-1)^2+\frac{a+1}{2(a-1)}+\frac{a+1}{2(a-1)}\ge 3\sqrt[3]{\frac{(a+1)^2}{2}}\ge 4.$$

The problem is solved and equality cannot be reached.

∇

Problem 40. *Find the best positive real constant k such that the following inequality holds for all positive real numbers a, b and c*

$$\frac{(a+b)(b+c)(c+a)}{abc}+\frac{k(ab+bc+ca)}{a^2+b^2+c^2}\ge 8+k.$$

(Pham Kim Hung)

SOLUTION. We clearly have

$$\frac{(a+b)(b+c)(c+a)}{abc}-8=\frac{c(a-b)^2+a(b-c)^2+b(c-a)^2}{abc},$$

$$1-\frac{ab+bc+ca}{a^2+b^2+c^2}=\frac{(a-b)^2+(b-c)^2+(c-a)^2}{2(a^2+b^2+c^2)}.$$

So we need to find a positive number k satisfying the condition

$$\sum_{sym}(a-b)^2\left(\frac{2(a^2+b^2+c^2)}{ab}-k\right)\ge 0 \Leftrightarrow \sum_{sym}(a-b)^2S_c\ge 0\ (\star)$$

where S_a, S_b, S_c are defined as

$$S_a = 2a(a^2+b^2+c^2)-kabc,$$

$$S_b = 2b(a^2+b^2+c^2)-kabc,$$

$$S_c = 2c(a^2+b^2+c^2)-kabc.$$

(*i*). *Necessary condition:* If $b = c$, we have $S_b = S_c$; so if $(\star)$ is true, we must have

$$S_b \geq 0 \Leftrightarrow 2(a^2 + 2b^2) \geq kab.$$

By **AM-GM** inequality, we find that the best value of k is $4\sqrt{2}$.

(*ii*). *Sufficient condition:* For $k \leq 4\sqrt{2}$, we will prove that the inequality is always true. WLOG, we may assume that $a \geq b \geq c$. Then $S_a \geq S_b \geq S_c$. Certainly, $S_a = 2a(a^2 + b^2 + c^2) - kabc \geq 0$. Let $x = \sqrt{bc}$, then

$$S_b + S_c = 2(b + c)(a^2 + b^2 + c^2) - 2kabc \geq$$

$$\geq 4x(a^2 + 2x^2) - 2kax^2 = 4x\left(a - \sqrt{2}x\right)^2 \geq 0.$$

We conclude that

$$\sum_{cyc} S_a(b - c)^2 \geq (S_b + S_c)(a - b)^2 \geq 0.$$

Conclusion: The best value of k is $4\sqrt{2}$. If $k = 4\sqrt{2}$, equality holds for $a = b = c$ or $a = \sqrt{2}b = \sqrt{2}c$ up to permutation. If $k < 4\sqrt{2}$, the equality holds only for $a = b = c$.

∇

Problem 41. *Suppose a, b, c are positive real numbers satisfying the condition $a + b + c + abc = 4$. Prove that*

$$\frac{a}{\sqrt{b + c}} + \frac{b}{\sqrt{c + a}} + \frac{c}{\sqrt{a + b}} \geq \frac{a + b + c}{\sqrt{2}}.$$

(Cezar Lupu)

SOLUTION. First we will prove that $a + b + c \geq ab + bc + ca$. Indeed, we may suppose that $c \geq b \geq a$ without loss of generality. We need to prove that

$$a + b - ab \geq \frac{4 - a - b}{ab + 1}(a + b - 1) \Leftrightarrow (a + b - 2)^2 \geq ab(a - 1)(b - 1).$$

Applying **AM-GM** inequality, we are done immediately

$$(a + b - 2)^2 \geq 4|(a - 1)(b - 1)| \geq ab|(a - 1)(b - 1)|.$$

Returning to our problem, **Cauchy-Schwarz** inequality yields that

$$c\sqrt{a + b} + a\sqrt{b + c} + b\sqrt{c + a} \leq \sqrt{2(a + b + c)(ab + bc + ca)},$$

and therefore

$$\frac{a}{\sqrt{b+c}}+\frac{b}{\sqrt{c+a}}+\frac{c}{\sqrt{a+b}} \geq \frac{(a+b+c)^2}{c\sqrt{a+b}+a\sqrt{b+c}+b\sqrt{c+a}}$$
$$\geq (a+b+c)\sqrt{\frac{a+b+c}{2(ab+bc+ca)}}$$
$$\geq \frac{a+b+c}{\sqrt{2}}.$$

This ends the proof. Equality holds for $a=b=c=1$.

∇

Problem 42. (*i*). *Prove that for all non-negative real numbers a,b,c, we have*

$$\sqrt{\frac{2a^2+bc}{a^2+2bc}}+\sqrt{\frac{2b^2+ca}{b^2+2ca}}+\sqrt{\frac{2c^2+ab}{c^2+2ab}} \geq 2\sqrt{2}.$$

(*ii*). *With the same condition, prove that*

$$\sqrt{\frac{a^2+2bc}{2a^2+bc}}+\sqrt{\frac{b^2+2ca}{2b^2+ca}}+\sqrt{\frac{c^2+2ab}{2c^2+ab}} \geq 2\sqrt{2}.$$

(Pham Kim Hung)

SOLUTION. (*i*). Since the inequality is homogeneous, we may assume that $abc=1$. The problem becomes

$$\sqrt{\frac{2x+1}{x+2}}+\sqrt{\frac{2y+1}{y+2}}+\sqrt{\frac{2z+1}{z+2}} \geq 2\sqrt{2},$$

where $x=a^3, y=b^3, z=c^2, xyz=1$. WLOG, suppose that $x \geq y \geq z$. Let $t=\sqrt{yz}$, then $t \leq 1$. First, notice that

$$\frac{(2y+1)(2z+1)}{(y+2)(z+2)}=\frac{4yz+2(y+z)+1}{yz+2(y+z)+4} \geq \frac{4t^2+4t+1}{t^2+4t+4}=\frac{(2t+1)^2}{(t+2)^2},$$

Therefore, applying **AM-GM** inequality, we obtain

$$\sqrt{\frac{2x+1}{x+2}}+\sqrt{\frac{2y+1}{y+2}}+\sqrt{\frac{2z+1}{z+2}} \geq \sqrt{\frac{2x+1}{x+2}}+2\sqrt{\frac{2t+1}{t+2}}=\sqrt{\frac{2+t^2}{2t^2+1}}+2\sqrt{\frac{2t+1}{t+2}}.$$

It suffices to prove that for all $t \leq 1$

$$\sqrt{(2+t^2)(2+t)}+2\sqrt{(2t+1)(2t^2+1)} \geq 2\sqrt{2(2t^2+1)(t+2)}.$$

After squaring both sides and reducing similar terms, we get an equivalent form

$$t^3+2t+4\sqrt{(2+t^2)(2+t)(1+2t)(1+2t^2)} \geq 22t^2+8.$$

Because $t \le 1$, $2t \ge 2t^2$. It is enough to prove that

$$\sqrt{(2+t^2)(2+t)(1+2t)(1+2t^2)} \ge 5t^2+2,$$

which is true by **Cauchy-Schwarz** inequality because $t+t^3 \ge 2t^2$ and

$$\sqrt{(2+t^2)(2+t)(1+2t)(1+2t^2)} \ge \sqrt{(4+5t^2)(1+5t^2)} \ge 2+5t^2.$$

This last step ends the proof.

(ii). This second part can be obtained from the first part by taking $bc = x^2, ca = y^2, ab = z^2$. Equality holds for $a = 0, b = c$ up to permutation.

∇

Problem 43. *Let x, y, z be non-negative real numbers with sum 1. Prove that*

$$\sqrt{x+\frac{(y-z)^2}{12}}+\sqrt{y+\frac{(z-x)^2}{12}}+\sqrt{z+\frac{(x-y)^2}{12}} \le \sqrt{3}.$$

(Phan Thanh Nam, VMEO 2004)

SOLUTION. Suppose $z = \min\{x, y, z\}$. First we will prove that if $u = y - z, v = x - z$ and $k = \frac{1}{12}$ then

$$\sqrt{x+ku^2}+\sqrt{y+kv^2} \le \sqrt{2(x+y)+k(u+v)^2}.$$

Indeed, this one is equivalent to

$$\begin{aligned}
&2\sqrt{(x+ku^2)(y+kv^2)} \le x+y+2kuv \\
\Leftrightarrow\ & 4(x+ku^2)(y+kv^2) \le (x+y+2kuv)^2 \\
\Leftrightarrow\ & (x-y)^2+4xkv(u-v)+4yku(v-u) \ge 0 \\
\Leftrightarrow\ & (x-y)^2+4k(u-v)(xv-yu) \ge 0 \\
\Leftrightarrow\ & (x-y)^2\left(1-4k(x+y-z)\right) \ge 0,
\end{aligned}$$

which is obvious. From the above result, we conclude that

$$\begin{aligned}
\text{LHS} &\le \sqrt{2(x+y)+\frac{(x+y-2z)^2}{12}}+\sqrt{z+\frac{(x-y)^2}{12}} \\
&= \sqrt{2(1-z)+\frac{(1-3z)^2}{12}}+\sqrt{z+\frac{(x-y)^2}{12}} \\
&\le \sqrt{2(1-z)+\frac{(1-3z)^2}{12}}+\sqrt{z+\frac{(1-3z)^2}{12}} \\
&= \frac{|5-3z|}{\sqrt{12}}+\frac{|1+3z|}{\sqrt{12}} = \sqrt{3},
\end{aligned}$$

which is exactly the desired result. The equality holds for $x = y = z = 1/3$.

Comment. With the same condition, we can prove that

$$\sqrt{x + (y-z)^2} + \sqrt{y + (z-x)^2} + \sqrt{z + (x-y)^2} \geq \sqrt{3}.$$

∇

Problem 44. *Let x, y, z be three non-negative real numbers satisfying the condition $xy + yz + zx = 1$. Prove that*

$$\frac{1}{\sqrt{x+y}} + \frac{1}{\sqrt{y+z}} + \frac{1}{\sqrt{z+x}} \geq 2 + \frac{1}{\sqrt{2}}.$$

(Le Trung Kien)

SOLUTION. WLOG, we may assume that $x = \max(x, y, z)$. Denote $a = y + z > 0$, then obviously, $ax = 1 - yz \leq 1$. Consider the function

$$\begin{aligned} f(x) &= \frac{1}{\sqrt{x+y}} + \frac{1}{\sqrt{y+z}} + \frac{1}{\sqrt{z+x}} \\ &= \frac{1}{\sqrt{y+z}} + \sqrt{\frac{2x + y + z + 2\sqrt{x^2+1}}{x^2+1}} \\ &= \frac{1}{\sqrt{a}} + \sqrt{\frac{2x + a + 2\sqrt{x^2+1}}{x^2+1}}. \end{aligned}$$

We have

$$f'(x) = \frac{yz - x^2 - x\sqrt{x^2+1}}{\sqrt{(x^2+1)^3(2x + a + 2\sqrt{x^2+1})}} \leq 0,$$

so $f(x)$ is a decreasing function. That means

$$\begin{aligned} f(x) \geq f\left(\frac{1}{a}\right) &= \sqrt{a} + \frac{1}{\sqrt{a}} + \sqrt{\frac{a}{a^2+1}} \\ &= (\sqrt{a} - 1)^2 \left(\frac{1}{\sqrt{a}} - \frac{(\sqrt{a}+1)^2}{2\sqrt{a(a^2+1)} + \sqrt{2}(a^2+1)} \right) + 2 + \frac{1}{\sqrt{2}}. \end{aligned}$$

Since

$$\frac{1}{\sqrt{a}} - \frac{(\sqrt{a}+1)^2}{2\sqrt{a(a^2+1)} + \sqrt{2}(a^2+1)} > 0,$$

and $f(x) \geq f\left(\frac{1}{a}\right) \geq 2 + \frac{1}{\sqrt{2}}$. We are done and equality holds for $x = y = 1, z = 0$ up to permutation.

∇

Problem 45. *Let a, b, c be positive real numbers. Prove that*

$$\left(2+\frac{a}{b}\right)^2+\left(2+\frac{b}{c}\right)^2+\left(2+\frac{c}{a}\right)^2 \ge \frac{9(a+b+c)^2}{ab+bc+ca}.$$

(Pham Kim Hung)

Solution. We can rewrite the inequality in the following form

$$\sum_{cyc}\frac{a^2}{b^2}+4\sum_{cyc}\frac{a}{b} \ge \frac{9(a^2+b^2+c^2)}{ab+bc+ca}+6.$$

Taking into account the following identities

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-3=\frac{(a-b)^2}{ab}+\frac{(c-a)(c-b)}{ac} ;$$

$$a^2+b^2+c^2-ab-bc-ca=(a-b)^2+(c-a)(c-b) ,$$

the inequality can be transformed into

$$(a-b)^2M+(c-a)(c-b)N \ge 0;$$

where

$$M=\frac{4}{ab}+\frac{(a+b)^2}{a^2b^2}-\frac{9}{ab+bc+ca} ;$$

$$N=\frac{4}{ac}+\frac{(c+a)(c+b)}{a^2c^2}-\frac{9}{ab+bc+ca} ;$$

Notice that if $a \ge b \ge c$ then

$$\sum_{cyc}\frac{a}{b} \le \sum_{cyc}\frac{b}{a} ; \sum_{cyc}\frac{a^2}{b^2} \le \sum_{cyc}\frac{b^2}{a^2} ; \qquad (\star)$$

So we only need to consider the case $a \ge b \ge c$, because the case $a \ge b \ge c$ will be reduced to their one after applying $(\star)$

$$N \ge \frac{5}{ac}+\frac{b}{ac^2}-\frac{9}{ab+bc+ca} > 0 ;$$

$$M+N \ge \frac{8}{ab}+\frac{5}{ac}-\frac{18}{ab+bc+ca} > 0 ;$$

Let $k=\frac{1+\sqrt{5}}{2}$. Consider the following subcases

(i). The first case. $a-b \le k(b-c)$. Then we have $(a-b)^2 \le (a-c)(b-c)$, so

$$(a-b)^2M+(c-a)(c-b)N \ge (a-b)^2(M+N) \ge 0.$$

(ii). The second case. $a-b \geq k(b-c)$. It suffices to prove that $M \geq 0$ or

$$(a^2+b^2+6ab)(ab+ca+cb) \geq 9a^2b^2.$$

Since $(k+1)b-a \leq kc$ and $k+1 = \dfrac{3+\sqrt{5}}{2}$, we deduce that

$$ab-(a-b)^2 = \left(\frac{(3+\sqrt{5})b}{2}-a\right)\left(a-\frac{(3-\sqrt{5})b}{2}\right) \leq kc\left(a+\frac{(3-\sqrt{5})b}{2}\right) \leq 2c(a+b).$$

Therefore

$$(a^2+b^2+6ab)(ab+ca+cb)-9a^2b^2 \geq ab\left((a^2-b^2)^2-ab\right)+c(a+b)^3$$

$$> c(a+b)^3-4abc(a+b) = c(a+b)(a-b)^2 \geq 0.$$

We have the conclusion. Equality holds for $a=b=c$.

Comment. In Mathematics and Youth Magazine, issue 4/2007, I proposed the following slightly simpler inequality

★ *Given positive real numbers a,b,c, prove that*

$$\left(1+\frac{2a}{b}\right)^2+\left(1+\frac{2b}{c}\right)^2+\left(1+\frac{2c}{a}\right)^2 \geq \frac{9(a+b+c)^2}{ab+bc+ca}.$$

∇

Problem 46. *Let a,b,c be non-negative real numbers and $a^2+b^2+c^2=3$. Prove that*

$$\frac{1}{3-ab}+\frac{1}{3-bc}+\frac{1}{3-ca}+\frac{1}{3-a^2}+\frac{1}{3-b^2}+\frac{1}{3-c^2} \geq 3.$$

(Pham Kim Hung)

SOLUTION. Rewrite the inequality in the form

$$\sum_{cyc}\left(\frac{3}{3-ab}-1\right)+\sum_{cyc}\left(\frac{3}{3-c^2}-1\right) \geq 3$$

$$\Leftrightarrow \sum_{cyc}\frac{ab}{3-ab}+\sum_{cyc}\frac{c^2}{3-c^2} \geq 3$$

Applying Cauchy-Schwarz, we obtain

$$\text{LHS} \geq \frac{(\sum ab+\sum c^2)^2}{3(\sum ab+\sum c^2)-\sum a^2b^2-\sum c^4}$$

Therefore, it suffices to prove that

$$\left(\sum_{cyc} ab+3\right)^2 \geq 3\left(3\sum_{cyc} ab-\sum_{cyc}(a^2)^2+\sum_{cyc} a^2b^2+3\sum a^2\right)$$

$$\Leftrightarrow \left(\sum_{cyc} ab\right)^2+6\left(\sum_{cyc} ab\right)+9\geq 9\left(\sum_{cyc} ab\right)+3\left(\sum_{cyc} a^2b^2\right)$$

$$\Leftrightarrow \left(\sum_{cyc} a^2\right)\left(\sum_{cyc} a^2-\sum_{cyc} ab\right)\geq \sum_{cyc} a^2(b-c)^2$$

$$\Leftrightarrow \sum_{cyc}(a^2+b^2-c^2)(a-b)^2\geq 0.$$

since $\sum a^2-\sum ab=\frac{1}{2}\sum(a-b)^2$.

Denote $S_a=b^2+c^2-a^2, S_b=c^2+a^2-b^2$ and $S_c=a^2+b^2-c^2$. Assume $a\geq b\geq c$, then $S_a\leq S_b\leq S_c$ and $(a-c)^2\geq(a-b)^2+(b-c)^2$. Also, $S_n\geq 0$ otherwise $ba^2\geq b^2>\frac{3}{2}$, false. We conclude that

$$\sum_{cyc}(a^2+b^2-c^2)(a-b)^2=\sum_{cyc} S_a(b-c)^2$$

$$\geq (S_a+S_b)(b-c)^2+(S_c+S_b)(a-b)^2=2a^2(a-b)^2+2c^2(b-c)^2\geq 0.$$

Equality holds for $a=b=c=1$ or $a=b=\sqrt{\frac{3}{2}}, c=0$ up to permutation.

∇

Problem 47. *Let a,b,c be three positive real numbers. Prove that*

$$\frac{1}{a\sqrt{a+b}}+\frac{1}{b\sqrt{b+c}}+\frac{1}{c\sqrt{c+a}}\geq \frac{3}{\sqrt{2abc}}.$$

(Phan Thanh Nam, VMEO 2005)

SOLUTION. Let $x=\frac{\sqrt{2bc}}{\sqrt{a(a+b)}}, y=\frac{\sqrt{2ca}}{\sqrt{b(b+c)}}, z=\frac{\sqrt{2ab}}{\sqrt{c(c+a)}}$. We need to prove that $x+y+z\geq 3$. However, it suffices to prove the following stronger result as follow

$$3\leq xy+yz+zx=\frac{2c}{\sqrt{(a+b)(b+c)}}+\frac{2a}{\sqrt{(b+c)(c+a)}}+\frac{2b}{\sqrt{(c+a)(a+b)}}.$$

Denote $u=\sqrt{b+c},\ v=\sqrt{c+a},\ w=\sqrt{a+b}$. We get

$$yz=\frac{w^2+v^2-u^2}{uv},\ zx=\frac{u^2+w^2-v^2}{vw},\ xy=\frac{v^2+u^2-w^2}{wu}.$$

The inequality becomes

$$v(v^2+u^2-w^2)+w(w^2+v^2-u^2)+u(u^2+w^2-v^2)\geq 3uvw$$

$$\Leftrightarrow (u^3+v^3+w^3)+(u^2v+v^2w+w^2u)\geq (v^2u+w^2v+u^2w)+3uvw.$$

But

$$(v^3+u^v)+(w^3+v^2w)+(u^3+w^2u)\geq 2(v^2u+w^2v+u^2w),$$
$$v^2u+w^2v+u^2w\geq 3uvw.$$

The proof is finished and equality holds iff $a=b=c$.

∇

Problem 48. *Prove that $ax+by+cz\geq 0$ if a,b,c,x,y,z are real numbers such that*

$$(a+b+c)(x+y+z)=3 \ ; \ (a^2+b^2+c^2)(x^2+y^2+z^2)=4.$$

(Mathlinks Contest)

SOLUTION. Let $\alpha=\sqrt[4]{\dfrac{a^2+b^2+c^2}{x^2+y^2+z^2}}$ and $a_1=\dfrac{a}{\alpha}, b_1=\dfrac{b}{\alpha}, c_1=\dfrac{c}{\alpha}, x_1=x\alpha, y_1=y\alpha, z_1=z\alpha$. We infer that

$$a_1^2+b_1^2+c_1^2=\frac{a^2+b^2+c^2}{\alpha^2}=\sqrt{(a^2+b^2+c^2)(x^2+y^2+z^2)}=2,$$
$$x_1^2+y_1^2+z_1^2=(x^2+y^2+z^2)\alpha^2=\sqrt{(a^2+b^2+c^2)(x^2+y^2+z^2)}=2,$$
$$a_1x_1+b_1y_1+c_1z_1=ax+by+cz.$$

The inequality can be rewritten as

$$(a_1+x_1)^2+(b_1+y_1)^2+(c_1+z_1)^2\geq 4.$$

According to the following relations

$$(a_1+b_1+c_1)(x_1+y_1+z_1)=(a+b+c)(x+y+z)=3,$$

we are done immediately since

$$(a_1+x_1)^2+(b_1+y_1)^2+(c_1+z_1)^2\geq \frac{1}{3}(a_1+b_1+c_1+x_1+y_1+z_1)^2$$
$$\geq \frac{4}{3}(a_1+b_1+c_1)(x_1+y_1+z_1)=4.$$

∇

Problem 49. *Let a, b, c, d be non-negative real numbers with sum 4. Prove that*

$$\sqrt{\frac{a+1}{ab+1}}+\sqrt{\frac{b+1}{bc+1}}+\sqrt{\frac{c+1}{cd+1}}+\sqrt{\frac{d+1}{da+1}} \ge 4.$$

(Pham Kim Hung)

SOLUTION. According to **AM-GM** inequality, we get

$$\text{LHS} \ge 4\sqrt[8]{\frac{(a+1)(b+1)(c+1)(b+1)}{(ab+1)(bc+1)(cd+1)(da+1)}},$$

and it remains to prove that

$$(a+1)(b+1)(c+1)(d+1) \ge (ab+1)(bc+1)(cd+1)(da+1).$$

After expanding, the inequality becomes

$$abcd + \sum_{sym} abc + \sum_{sym} ab + \sum_{sym} a + 1$$

$$\ge (abcd)^2 + abcd\sum_{cyc} ab + \sum_{cyc} ab^2c + \sum_{cyc} ab + 1 + 2abcd$$

$$\Leftrightarrow\ 4 + ac + bd + \sum_{sym} abc \ge (abcd)^2 + abcd + abcd\sum_{cyc} ab + \sum_{cyc} ab^2c.$$

The condition $a+b+c+d=4$ implies that $abcd \le 1$, therefore

$$ac + bd \ge 2\sqrt{abcd} \ge 2abcd \ge 2(abcd)^2 \ \Rightarrow ac + bd \ge abcd + (abcd)^2 \ (\star)$$

According to the inequality $(x+y+z+t)^2 \ge 4(xy+yz+zt+tx)$, we obtain

$$16 = \left(\sum_{cyc} a\right)^2 \ge 4\sum_{cyc} ab \ \Rightarrow \sum_{cyc} ab \le 4$$

$$\Rightarrow 16 \ge \left(\sum_{cyc} ab\right)^2 \ge 4\sum_{cyc} ab^2c \ \Rightarrow 4 \ge \sum_{cyc} ab^2c \ (\star\star)$$

Moreover, we also have

$$\left(\sum_{cyc} abc\right)^2 \ge 4abcd\sum_{cyc} ab \ \Rightarrow \sum_{cyc} abc \ge abcd\sum_{cyc} ab \ (\star\star\star).$$

Using $(\star), (\star\star)$ and $(\star\star\star)$, we get the conclusion immediately.

∇

Problem 50. *Prove that for all non-negative real numbers a, b, c then*

$$\frac{a}{\sqrt{a+b}}+\frac{b}{\sqrt{b+c}}+\frac{c}{\sqrt{c+a}} \geq \frac{\sqrt{a}+\sqrt{b}+\sqrt{c}}{\sqrt{2}}.$$

SOLUTION. Let $x=\sqrt{a},\ y=\sqrt{b},\ z=\sqrt{c}$. The inequality becomes

$$\frac{x^2}{\sqrt{x^2+y^2}}+\frac{y^2}{\sqrt{y^2+z^2}}+\frac{z^2}{\sqrt{z^2+x^2}} \geq \frac{x+y+z}{\sqrt{2}}$$

$$\Leftrightarrow \sum_{cyc}\frac{2x^4}{x^2+y^2}+\sum_{cyc}\frac{4x^2y^2}{\sqrt{(x^2+y^2)(y^2+z^2)}} \geq (x+y+z)^2.$$

Notice that

$$\frac{x^4-y^4}{x^2+y^2}+\frac{y^4-z^4}{y^2+z^2}+\frac{z^4-x^4}{z^2+x^2}=0,$$

hence

$$\frac{2x^4}{x^2+y^2}+\frac{2y^4}{y^2+z^2}+\frac{2z^4}{z^2+x^2}=\frac{x^4+y^4}{x^2+y^2}+\frac{y^4+z^4}{y^2+z^2}+\frac{z^4+x^4}{z^2+x^2}.$$

Furthermore, the following sequences

$$\left(\frac{x^2y^2}{\sqrt{x^2+y^2}},\frac{y^2z^2}{\sqrt{y^2+z^2}},\frac{z^2x^2}{\sqrt{z^2+x^2}},\right) \quad ; \quad \left(\frac{1}{\sqrt{x^2+y^2}},\frac{1}{\sqrt{y^2+z^2}},\frac{1}{\sqrt{z^2+x^2}}\right).$$

are monotone in the opposite order, so **Rearrangement** inequality shows that

$$\sum_{cyc}\frac{4x^2y^2}{\sqrt{(x^2+y^2)}}\cdot\frac{1}{\sqrt{(y^2+z^2)}} \geq \sum_{sym}\frac{4x^2y^2}{\sqrt{x^2+y^2}}\cdot\frac{1}{\sqrt{x^2+y^2}}$$

$$\Rightarrow \sum_{cyc}\frac{4x^2y^2}{\sqrt{(x^2+y^2)(y^2+z^2)}} \geq \sum_{sym}\frac{4x^2y^2}{x^2+y^2}.$$

It remains to prove that

$$\sum_{cyc}\frac{x^4+y^4}{x^2+y^2}+\sum_{cyc}\frac{4x^2y^2}{x^2+y^2} \geq (x+y+z)^2$$

$$\Leftrightarrow \sum_{cyc}\frac{x^2+y^2}{2}+\sum_{cyc}\frac{2x^2y^2}{x^2+y^2} \geq 2\sum_{cyc}xy$$

which is obvious. Equality holds for $x=y=z$, or equivalently $a=b=c$.

∇

Problem 51. *Let a, b, c be real numbers. Prove that*

$$\sqrt[3]{2a^2-bc}+\sqrt[3]{2b^2-ca}+\sqrt[3]{2c^2-ab} \geq 0.$$

(Pham Kim Hung)

SOLUTION. First, we notice that the inequality only needs to be considered for a, b, c non-negative. Now, consider the identity

$$x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$$

therefore, $(x + y + z)(x^3 + y^3 + z^3 - 3xyz) \geq 0$, so we can write

$$\sqrt[3]{2a^2 - bc} + \sqrt[3]{2b^2 - ca} + \sqrt[3]{2c^2 - ab} \geq 0$$

$$\Leftrightarrow\ 2\sum_{cyc} a^2 - \sum_{cyc} bc \geq 3\sqrt[3]{(2a^2 - bc)(2b^2 - ca)(2c^2 - ab)}\ (\star)$$

Without loss of generality, assume that $a \geq b \geq c$. Notice that the inequality is obvious if $\sqrt[3]{2b^2 - ca} \geq 0$, $\sqrt[3]{2c^2 - ab} \geq 0$. It is also obvious if $(2b^2 - ca)(2c^2 - ab) \leq 0$ (due to $(\star)$), so we may assume that $2b^2 - ca \leq 0, 2c^2 - ab \leq 0$.

(ii). The first case. $a \geq 2(b + c)$. We conclude that

$$2a^2 - bc \geq 4(ab - 2c^2 + ac - 2b^2)$$

$$\Rightarrow\ \sqrt[3]{2a^2 - bc} \geq -\sqrt[3]{2b^2 - ca} - \sqrt[3]{2c^2 - ab}$$

$$\Rightarrow\ \sqrt[3]{2a^2 - bc} + \sqrt[3]{2b^2 - ca} + \sqrt[3]{2c^2 - ab} \geq 0.$$

We used the inequality $\sqrt[3]{4(x + y)} \geq \sqrt[3]{x} + \sqrt[3]{y}$.

(i). The second case. If $a < 2(b + c)$. WLOG, assume that $abc = 1$. We have to prove that

$$2(a^2 + b^2 + c^2) - (ab + bc + ca) \geq 3\sqrt[3]{(2a^3 - 1)(1 - 2b^3)(1 - 2c^3)}.$$

Denote

$$f(a, b, c) = 2(a^2 + b^2 + c^2) - (ab + bc + ca) - 3\sqrt[3]{(2a^3 - 1)(1 - 2b^3)(1 - 2c^3)}.$$

We clearly have $f(a, b, c) \geq f(a, \sqrt{bc}, \sqrt{bc})$ since

$$2(a^2 + b^2 + c^2) - (ab + bc + ca) \geq 2(a^2 + 2bc) - (2a\sqrt{bc} + bc)\ ;$$

and

$$(1 - 2b^3)(1 - 2c^3) \leq (1 - 2\sqrt{b^3c^3})^2\ ;$$

It remains to prove the initial inequality in the case $b = c$, namely

$$\sqrt[3]{2a^2 - b^2} \geq 2\sqrt[3]{ab - 2b^2}.$$

This inequality is equivalent to

$$2a^2 + 15b^2 \geq 9ab,$$

which is clearly true due to **AM-GM** inequality. Equality holds for $a = b = c = 0$.

Comment. The following stronger inequality holds

★ *Given real numbers a, b, c, and $k = \dfrac{1+\sqrt{513}}{16}$, then*

$$\sqrt[3]{ka^2 - bc} + \sqrt[3]{kb^2 - ca} + \sqrt[3]{kc^2 - ab} \geq 0.$$

To prove it, we use the same technique as shown in the above proof. Similarly, we only need to consider the main case $a \geq b \geq c, abc = 1, kb^3 \leq 1$ and $kc^3 \leq 1$. Let

$$f(a,b,c) = k(a^2+b^2+c^2) - (ab+bc+ca) - 3\sqrt[3]{(ka^3-1)(1-kb^3)(1-kc^3)}.$$

We are done easily if $a \leq k(\sqrt{b} + \sqrt{c})^2$ since, in this case, we have $f(a,b,c) \geq f(a, \sqrt{bc}, \sqrt{bc})$. It remains to consider the case $a \geq k(\sqrt{b} + \sqrt{c})^2$. Denote

$$g(a) = ka^2 - bc + 4(kb^2 + kc^2 - ab - ac).$$

We infer that

$$g'(a) = 2ka - 4k(b+c) \geq 2k^2\left(\sqrt{b} + \sqrt{c}\right)^2 - 4(b+c) \geq 0.$$

Therefore

$$g(a) \geq g\left(k\left(\sqrt{b} + \sqrt{c}\right)^2\right).$$

Denote $x = \sqrt{b}, y = \sqrt{c}$. The inequality $g\left(k\left(\sqrt{b} + \sqrt{c}\right)^2\right) \geq 0$ is equivalent to

$$k^3(x+y)^4 - x^2y^2 + 4k(x^4+y^4) - 4k(x+y)^2(x^2+y^2) \geq 0$$

or

$$k^3(x^4+y^4) + (4k^3-8k)(x^3y+xy^3) + (6k^3-8k-1)x^2y^2 \geq 0.$$

This last inequality is obvious since all the coefficients are non-negative. Therefore

$$g(a) = ka^2 - bc + 4(kb^2 + kc^2 - ab - ac) \geq 0$$

$$\Rightarrow \sqrt[3]{ka^2 - bc} \geq \sqrt[3]{4(ab + ac - kb^2 - kc^2)} \geq \sqrt[3]{ab - kc^2} + \sqrt[3]{ac - kb^2}$$

$$\Rightarrow \sqrt[3]{ka^2 - bc} + \sqrt[3]{kb^2 - ca} + \sqrt[3]{kc^2 - ab} \geq 0,$$

which is the desired result. Equality holds for $(a,b,c) \sim \left(\sqrt{\dfrac{8k-1}{k}}, 1, 1\right)$.

∇

Problem 52. *Prove that the following inequality holds for all real numbers a, b, c*

$$(a^2+b^2+c^2)^2 \geq 3(a^3b+b^3c+c^3a).$$

(Vasile Cirtoaje)

SOLUTION. We give four solutions to this problem.

First Solution. Notice that

$$\begin{aligned} &4(a^2+b^2+c^2-ab-bc-ca)\Big((a^2+b^2+c^2)^2-3(a^3b+b^3c+c^3a)\Big)\\ &=\Big((a^3+b^3+c^3)-5(a^2b+b^2c+c^2a)+4(ab^2+bc^2+ca^2)\Big)^2\\ &\quad+3\Big((a^3+b^3+c^3)-(a^2b+b^2c+c^2a)-2(ab^2+bc^2+ca^2)+6abc\Big)^2 \geq 0. \end{aligned}$$

Second Solution. WLOG, suppose that $a = \min(a, b, c)$. Let $b = a + x, c = a + y$ with $x, y \geq 0$. By expanding, we obtain

$$(a^2+b^2+c^2)^2-3(a^3b+b^3c+c^3a)=$$

$$=(x^2+y^2-xy)a^2+(x^3+y^3+4xy^2-5x^2y)a+x^4+y^4+2x^2y^2-3x^3y.$$

Consider the expression as a quadratic of a, then

$$\begin{aligned} \Delta_f &= (x^3+y^3+4xy^2-5x^2y)-4(x^2+y^2-xy)(x^4+y^4+2x^2y^2-3x^3y)\\ &=-3(x^3-x^2y-2xy^2-y^3)^2 \leq 0, \end{aligned}$$

so the desired result follows immediately.

Third Solution. The following identity gives the conclusion

$$2(a^2+b^2+c^2)^2-6(a^3b+b^3c+c^3a)=\sum_{cyc}(a^2-2ab+bc-c^2+ca)^2.$$

Fourth Solution. The following identity gives the conclusion

$$6(a^2+b^2+c^2)^2-12(a^3b+b^3c+c^3a)=\sum_{cyc}(a^2-2b^2+c^2+3bc-3ca)^2.$$

Comment. Using this result, we can prove the following inequality easily enough by the *Cauchy reverse* techique.

★ *Let x, y, z be positive real numbers such that $x + y + z = 3$. Prove that*

$$\frac{x}{1+xy}+\frac{y}{1+yz}+\frac{z}{1+zx} \geq \frac{3}{2}.$$

Indeed, to prove this inequality, just notice that

$$\frac{x}{1+xy} = x - \frac{x^2y}{1+xy} \geq x - \frac{x^2y}{2\sqrt{xy}} = x - \frac{1}{2}\sqrt{x^3y}.$$

∇

Problem 53. *Let a, b, c be three real numbers satisfying the condition $a^2+b^2+c^2 = 9$. Prove that*

$$3\min(a, b, c) \leq 1 + abc.$$

(Virgil Nicula)

SOLUTION. WLOG, we may assume that $c \geq b \geq a$. Consider the following cases

(i). $a \leq 0$: Let $d = -a$ and $e = |b|$. We will prove that

$$-3d \leq 1 - dce \Leftrightarrow d(ce-3) \leq 1.$$

If $ce \leq 3$, the conclusion follows immediately. Otherwise, if $ce \geq 3$ then

$$d^2(ce-3)(ce-3) \leq \left(\frac{d^2+2ce-6}{3}\right)^3 \leq \left(\frac{d^2+c^2+e^2-6}{3}\right)^3 = 1,$$

and we are done. Equality holds for $a = -1, b = c = 2$ up to permutation.

(ii). If $a \geq 0$: The problem can be rewritten as

$$a(a^2+b^2+c^2) \leq 3 + 3abc.$$

Since $2abc \geq a^3 + ab^2$, we only need to prove

$$3 + abc \geq ac^2 \Leftrightarrow 3 \geq ac(c-b).$$

$a \leq b$, hence $c \leq \sqrt{9-2a^2}$, so

$$ac(c-b) \leq ac(c-a) \leq a\sqrt{9-2a^2}(\sqrt{9-2a^2}-a).$$

It suffices to prove that

$$a(9-2a^2) - a^2\sqrt{9-2a} \leq 3$$

$$\Leftrightarrow f(a) = 2a^6 - 9a^4 - (3a-1)^2 \leq 0.$$

If $\frac{1}{3} \leq 2 \leq 1$ then

$$f(a) = 2a^4(a^2-1) - 7a^4 - (3a-1)^2 \leq 0.$$

If $1 \le a \le \sqrt{\frac{3}{2}}$ then

$$f(a) = \left(a^4 + \frac{3}{2}\right)(2a^2 - 3) - 6a^2(a^2 - 1) - 6a\left(a - \frac{11}{12}\right) \le 0.$$

If $\sqrt{\frac{3}{2}} \le a \le \sqrt{3}$ then

$$f(a) = a^2(a^2 - 3)(2a^2 - 3) + (1 - 6a) \le 0.$$

The problem is completely solved. There is just one case of equality.

Comment. The following inequality, proposed by Vasile Cirtoaje, can be proved in the same manner.

★ *Given non-negative real numbers a, b, c with $a^2 + b^2 + c^2 = 3$, prove that*

$$1 + 4abc \ge 5 \min\{a, b, c\}.$$

∇

Problem 54. *Let a, b, c, d be non-negative real numbers such that $a + b + c + d = 4$. Prove that*

$$(1 + a^4)(1 + b^4)(1 + c^4)(1 + d^4) \ge (1 + a^3)(1 + b^3)(1 + c^3)(1 + d^3).$$

(Pham Kim Hung)

SOLUTION. Notice that for all $x \ge 0$, $(1 + x^4)(1 + x) \ge (1 + x^3)(1 + x^2)$, therefore

$$\prod_{cyc}(1 + a^4)\prod_{cyc}(1 + a) \ge \prod_{cyc}(1 + a^3)\prod_{cyc}(1 + a^2).$$

It's enough to prove that $\prod_{cyc}(1 + a^2) \ge \prod_{cyc}(1 + a)$, or $\sum_{cyc}\ln(1 + a^2) \ge \sum_{cyc}\ln(1 + a)$. Denote

$$f(x) = \ln(1 + a^2) - \ln(1 + a) - \frac{a - 1}{2}.$$

Its derivative is

$$f'(x) = \frac{2x}{1 + x^2} - \frac{1}{1 + x} - \frac{1}{2} = \frac{(x - 1)(3 - x^2)}{2(1 + x)(1 + x^2)},$$

so $f(x)$ is increasing on $[1, \sqrt{3}]$ and decreasing on $[0, 1] \cup [\sqrt{3}, +\infty]$. That implies

$$\min_{0 \le x \le 2.2} f(x) = \min\{f(1), f(2.2)\} = 0.$$

If all a, b, c, d are smaller than 2.2 then we conclude that

$$\sum_{cyc} f(a) \geq 0 \Rightarrow \sum_{cyc} \ln(1+a^2) - \ln(1+a) \geq \sum_{cyc} \frac{a-1}{2} = 0.$$

Otherwise, suppose $a \geq 2.2$. Since the function $g(x) = \dfrac{1+x^2}{1+x}$ attains its minimum on $\mathbb{R}^+$ for $x = -1+\sqrt{2}$ and $g(a) \geq g(2.2)$, we deduce that

$$g(a) \cdot g(b) \cdot g(c) \cdot g(d) \geq g(2.2) \cdot \left(g\left(-1+\sqrt{2}\right)\right)^3 \approx 1.03 > 1.$$

This ends the proof and equality holds for $a = b = c = d = 1$.

∇

Problem 55. *Find the best constant k (smallest) for the inequality*

$$a^k + b^k + c^k \geq ab + bc + ca,$$

to hold for a, b, c are three non-negative real numbers with $a+b+c=3$.

(Generalization of Russia MO 2000)

SOLUTION. In example 1.1.1 in chapter I, this inequality is proved for $k = \frac{1}{2}$ and therefore it's true for every $k \geq \frac{1}{2}$. Consider the inequality in the case $k \leq \frac{1}{2}$.

Lemma. *Suppose $a, b \geq 0$ and $a+b = 2t \geq 1$ then we have*

$$a^k + b^k - ab \geq \min\left((2t)^k, 2t^k - t^2\right).$$

Indeed, WLOG assume that $a \geq b$. There exists a non-negative real number x with $a = t+x, b = t-x$. Consider the function

$$f(x) = (t+x)^k + (t-x)^k - t^2 + x^2$$

then

$$\begin{aligned}
f'(x) &= k(t+x)^{k-1} - k(t-x)^{k-1} + 2x, \\
f''(x) &= k(k-1)\left((t+x)^{k-2} + (t-x)^{k-2}\right) + 2, \\
f'''(x) &= k(k-1)(k-2)\left((t+x)^{k-3} - (t-x)^{k-3}\right).
\end{aligned}$$

So $f'''(x) \leq 0$, hence $f''(x)$ is a monotone function and therefore $f'(x)$ has no more than two roots. Since $f'(0) = 0$ and

$$f''(0) = 2k(k-1)t^{k-2} + 2 = 2 - 2k(1-k) \geq 0,$$

we conclude that f attains the minimum at $x=0$ or $x=t$ only.

Returning to our problem, WLOG, assume that $a \geq b \geq c$. Let $a+b=2t \geq 1$, then

$$a^k+b^k+c^k-(ab+bc+ca) \geq \min\left\{(2t)^k, 2t^k-t^2\right\}-2ct+c^k.$$

(i). $(2t)^k \leq 2t^k-t^2$. Using the lemma for $2t$ and c, we obtain

$$a^k+b^k+c^k-(ab+bc+ca) \geq (2t)^k+c^k-c\cdot 2t \geq \min\left\{(2t+c)^k, 2\left(t+\frac{c}{2}\right)^k-\left(t+\frac{c}{2}\right)^2\right\}.$$

Since $2t+c=3$, we can conclude that

$$a^k+b^k+c^k-(ab+bc+ca) \geq \min\left\{3^k, 2\cdot\frac{3^k}{2^k}-\frac{9}{4}\right\}.$$

(ii). $(2t)^k \geq 2t^k-t^2$. We will prove that $g(t) \geq 0$ where

$$g(t)=2t^k+(3-2t)^k-2t(3-2t)+t^2=2t^k+(3-2t)^k-6t+3t^2.$$

Notice that

$$\begin{aligned}
g'(t) &= 2kt^{k-1}-2k(3-2t)^{k-1}-6+6t,\\
g''(t) &= 2k(k-1)\left(t^{k-2}-2(3-2t)^{k-2}\right)+6,\\
g'''(t) &= 2k(k-1)(k-2)\left(t^{k-3}-4(3-2t)^{k-3}\right).
\end{aligned}$$

Because $g'''(t)$ has no roots if $(t \geq 1)$, we infer $g'(t)$ has no more than two roots. We deduce that

$$\min_{1\leq t\leq 3/2} g(t)=\min\left(g(1), g\left(\frac{3}{2}\right)\right)=\min\left(0, 2\cdot\frac{3^k}{2^k}-\frac{9}{4}\right).$$

According to these results, we conclude that for all positive real k

$$a^k+b^k+c^k-(ab+bc+ca) \geq \min\left(0, 2\cdot\frac{3^k}{2^k}-\frac{9}{4}\right).$$

Therefore the best constant k is

$$2\cdot\frac{3^k}{2^k}=\frac{9}{4} \Leftrightarrow k=\frac{2\ln 3-3\ln 2}{\ln 3-\ln 2} \approx 0.2905.$$

In this case, equality holds for $a=b=c=1$ or $a=b=\frac{3}{2}, c=0$ or permutations.

∇

Problem 56. *Let a, b, c be positive real numbers. Prove that*

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq 3\sqrt{\frac{a^2+b^2+c^2}{ab+bc+ca}}.$$

(Vo Quoc Ba Can)

Solution. Notice that if $a \geq b \geq c$ then

$$\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)-\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)=\frac{(a-b)(a-c)(c-b)}{abc}\leq 0$$

so it's enough to consider the case $a \geq b \geq c$. By squaring both sides, we get

$$\sum_{cyc}\frac{a^2}{b^2}+\sum_{cyc}\frac{2b}{a}\geq\frac{9(a^2+b^2+c^2)}{ab+bc+ca}.$$

Moreover, using the following identities

$$\frac{b}{a}+\frac{c}{b}+\frac{a}{c}-3=\frac{(b-c)^2}{bc}+\frac{(a-b)(a-c)}{ac}$$

$$\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}-3=\frac{(b-c)^2(b+c)^2}{b^2c^2}+\frac{(a^2-b^2)(a^2-c^2)}{a^2b^2}$$

and $a^2+b^2+c^2-(ab+bc+ca)=(b-c)^2+(a-b)(a-c)$, we can rewrite this inequality to $(b-c)^2M+(a-b)(a-c)N \geq 0$ with

$$M=\frac{2}{bc}+\frac{(b+c)^2}{b^2c^2}-\frac{9}{ab+bc+ca}\ ;$$

$$N=\frac{2}{ac}+\frac{(a+b)(a+c)}{a^2b^2}-\frac{9}{ab+bc+ca}\ ;$$

If $b-c \geq a-b$ then $2(b-c)^2 \geq (a-b)(a-c)$. We have

$$M \geq \frac{6}{bc}-\frac{9}{ab+bc+ca} \geq 0\ ;$$

$$M+2N \geq \frac{6}{bc}-\frac{18}{ab+bc+ca} \geq 0\ ;$$

We conclude that

$$M(b-c)^2+N(a-b)(a-c) \geq \frac{1}{2}(a-b)(a-c)(M+2N) \geq 0.$$

Now suppose that $b-c \leq a-b$, then $2b \leq a+c$. Certainly $M \geq 0$ and

$$N \geq \frac{2}{ac}+\frac{a+b+c}{ab^2} \geq \frac{2}{ac}+\frac{3}{ab} \geq \frac{(\sqrt{2}+\sqrt{3})^2}{ac+ab} > \frac{9}{ab+bc+ca}.$$

This ends the proof. Equality holds for $a=b=c$.

Comment. The following similar inequality is a bit more difficult

★ *Given positive real numbers a, b, c, prove that*

$$\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}+\frac{9(ab+bc+ca)}{a^2+b^2+c^2} \geq 12.$$

PROOF. Notice that if $a \geq b \geq c$ then

$$\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2} \geq \frac{b^2}{a^2}+\frac{c^2}{b^2}+\frac{a^2}{c^2},$$

so we only need to consider the inequality in the case $a \geq b \geq c$. Rewrite it to

$$(a-b)^2\left(\frac{(a+b)^2}{a^2b^2}-\frac{9}{a^2+b^2+c^2}\right)+(c-a)(c-b)\left(\frac{(c+a)(c+b)}{a^2c^2}-\frac{9}{a^2+b^2+c^2}\right) \geq 12,$$

then we denote

$$M=\frac{(a+b)^2}{a^2b^2}-\frac{9}{a^2+b^2+c^2}\ ;\ N=\frac{(c+a)(c+b)}{a^2c^2}-\frac{9}{a^2+b^2+c^2}.$$

First we will prove that $N \geq 0$, or

$$(c+a)(c+b)(a^2+b^2c^2) \geq 9c^2a^2.$$

Since $b \geq c$, we may prove the following stronger inequality as

$$(c+a)(2c)(a^2+2c^2) \geq 9c^2a^2$$

or

$$2a^3-7a^2c+4c^2a+4c^3 \geq 0$$

or

$$(a-c)^2(2a+c) \geq 0.$$

which is obvious, so $N \geq 0$. Next, we divide the problem in two cases.

(i). The first case. $a-b \leq b-c$, then

$$(c-a)(c-b) \geq 2(a-b)^2$$

and the inequality is proved if we can prove that

$$M+2N \geq \frac{(c+a)(c+b)}{a^2c^2}-\frac{10}{a^2+b^2+c^2} \geq 0.$$

Indeed, this inequality is equivalent to

$$(c+a)(c+b)(a^2+b^2+c^2) \geq 10a^2c^2,$$

Since $b \geq \dfrac{a+c}{2}$, we infer that (due to **AM-GM** inequality)

$$(c+a)(c+b)(a^2+b^2+c^2) \geq (c+a)\left(c+\frac{a+c}{2}\right)\left(a^2+c^2+\frac{1}{4}(a+c)^2\right)$$

$$\geq 2\sqrt{ac}\cdot\sqrt{3ac}\cdot 3ac > 10ac.$$

(ii). The second case. $a-b\geq b-c$. In this case, we will prove that $M\geq 0$ or

$$(a+b)^2(a+b+c)^2\geq 9a^2b^2.$$

If $a\geq 2b$, this inequality is immediately true because $a^2+b^2\geq \frac{5}{2}ab$. So we may assume that $a\leq 2b$. Since $c\geq 2b-a$, we only need to prove that

$$(a+b)^2\left(a^2+b^2+(2b-a)^2\right)^2\geq 9a^2b^2.$$

Let $x=\frac{b}{a}$, then we have $\frac{1}{2}\leq x\leq 1$. The inequality becomes

$$(x+1)^2(5x^2-4x+2)\geq 9x^2$$

or

$$f(x)=5x^4+6x^3-10x^2+2\geq 0.$$

The derivative $f'(x)=20x^3+18x^2-20x$ has exactly one root in $\left[\frac{1}{2},1\right]$, $x_0=\frac{-9+\sqrt{481}}{40}$. Therefore $\min\limits_{0.5\leq x\leq 1} f(x)=f(x_0)>0$, and the conclusion follows.

∇

Problem 57. *Suppose that a,b,c,d are positive real numbers satisfying $a^2+b^2+c^2+d^2=4$. Prove that*

$$\frac{1}{3-abc}+\frac{1}{3-bcd}+\frac{1}{3-cda}+\frac{1}{3-dab}\leq 2.$$

(Pham Kim Hung)

SOLUTION. Let $x=abc, y=abd, z=acd, t=bcd$. The problem becomes

$$\sum_{cyc}\frac{1}{3-x}\leq 2 \Leftrightarrow \sum_{cyc}\frac{1-x}{3-x}\geq 0.$$

According to AM-GM inequality, we deduce

$$x+y=ab(c+d)\leq\frac{1}{2}(a^2+b^2)\sqrt{2(c^2+d^2)}\leq\left(\frac{4}{3}\right)^{3/2}<\frac{9}{4}.$$

WLOG, assume that $x\leq y\leq z\leq t$. First, we consider the case $x+y\leq\frac{1}{4}$. In this case, it's easy to see that $t=bcd\leq\left(\frac{4}{3}\right)^{3/2}$ and $z\leq 1$. Since $(3-x)^{-1}$ is an increasing convex function, we have

$$\frac{1}{3-x}+\frac{1}{3-y}+\frac{1}{3-z}+\frac{1}{3-t}\leq\frac{1}{3-1/4}+\frac{1}{3}+\frac{1}{3-1}+\frac{1}{3-(4/3)^{3/2}}<2.$$

Now suppose that $x+y \ge \frac{1}{4}$. Since

$$(1-x)(4x+3)-(1-y)(4y+3)=(x-y)(1-4x-4y)\ge 0,$$
$$(3-x)(4x+3)-(3-y)(4y+3)=(x-y)(9-4x-4y)\le 0,$$

according to Chebyshev inequality, we deduce that

$$\sum_{cyc}\frac{1-x}{3-x}=\sum_{cyc}\frac{(1-x)(4x+3)}{(3-x)(4x+3)}\ge\frac{1}{4}\left(\sum_{cyc}(1-x)(4x+3)\right)\left(\sum_{cyc}\frac{1}{(3-x)(4x+3)}\right).$$

It remains to prove that $\sum_{cyc}(1-x)(4x+3)\ge 0$ or $S=12+\sum_{cyc}x-4\sum_{cyc}x^2\ge 0$.

Since $\sum_{cyc}a^2=4$, AM-GM inequality shows that $abcd\le 1$ and $\sum_{cyc}\frac{1}{a}\ge 4$. Therefore

$$x+y+z+t=abcd\left(\sum_{cyc}\frac{1}{a}\right)\ge 4abcd\ge 4a^2b^2c^2d^2 \quad (\star)$$

Let $m=a^2, n=b^2, p=c^2, q=d^2$ then $\sum_{cyc}m=4$. According to $(\star)$, we infer that

$$S\ge 12+4mnpq-4\sum_{cyc}mnp$$

Since $x\le y\le z\le t$, it follows that $m\le n\le p\le q$. Let $r=\frac{1}{3}(n+p+q)$ then by AM-GM inequality, we get $np+pq+qn\le 3r^2$ and $npq\le t^3$. Because $m\le 1$, we get

$$S=12-4npq(1-m)-4m(np+pq+qm)\ge 12-4r^3(1-m)+4m\cdot 3r^2.$$

Replacing m with $4-3r$ in the inequality above, we obtain

$$S\ge 4-4(4-3r)r^2-4r^3(3r-3)=12(r-1)^2(-r^2+2r+1)\ge 0.$$

The problem is completely solved and reaches equality for $a=b=c=d=1$.

∇

Problem 58. *Suppose that n positive real numbers $x_1, x_2, ..., x_n$ satisfy the condition*

$$\frac{1}{1+x_1}+\frac{1}{1+x_2}+...+\frac{1}{1+x_n}=\frac{n}{2}.$$

Prove that

$$\sum_{i,j=1}^{n}\frac{1}{x_i+x_j}\ge\frac{n^2}{2}.$$

(Titu Andreescu, Gabriel Dospinescu)

SOLUTION. For each $i \in \{1, 2, ..., n\}$, we denote $a_i = \dfrac{1-x_i}{1+x_i}$. Therefore, $a_1 + a_2 + ... + a_n = 0$ and $a_i \in [-1, 1]$. Consider the expression

$$S = \sum_{i,j=1}^{n} \frac{1}{x_i + x_j} \Rightarrow 2S = \sum_{i,j=1}^{n} \frac{(1+a_i)(1+a_j)}{1 - a_i a_j}.$$

We have

$$P = \sum_{i,j=1}^{n} (1+a_i)(1+a_j)(1-a_i a_j) = n^2 - \sum_{i,j=1}^{n} a_i^2 a_j^2 \le n^2.$$

According to **Cauchy-Schwarz** inequality, we conclude

$$2S \cdot P \ge \left(\sum_{i,j=1,n} (1+a_i)(1+a_j) \right)^2 = n^4.$$

Therefore $S \ge \dfrac{n^2}{2}$ and the equality holds for $x_1 = x_2 = ... = x_n = 1$.

∇

Problem 59. *Let a, b, c be non-negative real numbers. Prove that*

$$\frac{ab}{a+4b+4c} + \frac{bc}{b+4c+4a} + \frac{ca}{c+4a+4b} \le \frac{a+b+c}{9}.$$

(Pham Kim Hung)

SOLUTION. WLOG, we may assume that $a + b + c = 3$. The inequality becomes

$$\sum_{cyc} \frac{3ab}{a + 4(3-a)} \le 1 \Leftrightarrow \sum_{cyc} \frac{ab}{4-a} \le 1 \Leftrightarrow \sum_{cyc} \frac{b}{4-a} \le 1$$

$$\Leftrightarrow \sum_{cyc} b(4-b)(4-c) \le \prod_{cyc} (4-a) \Leftrightarrow a^2b + b^2c + c^2a + abc \le 4.$$

We have two different solutions, for this last inequality

First Solution. Since $a + b + c = 3$, the inequality can be rewritten to

$$27(a^2b + b^2c + c^2a) + 27abc \le 4(a+b+c)^3$$

$$\Leftrightarrow 27 \sum_{cyc} a^2b + 27abc \le 4 \sum_{cyc} a^3 + 12 \sum_{cyc} a^2b + 12 \sum_{cyc} ab^2 + 24abc$$

$$\Leftrightarrow 15 \sum_{cyc} a^2b + 3abc \le 4 \sum_{cyc} a^3 + 12 \sum_{cyc} ab^2$$

$$\Leftrightarrow\ 12\left(\sum_{cyc}a^2b-\sum_{cyc}ab^2\right)\le 4\left(\sum_{cyc}a^3-\sum_{cyc}a^2b\right)+\left(-3abc+\sum_{cyc}a^3\right).$$

This inequality can be rewritten to

$$12(a-b)(a-c)(b-c)\le S_a(b-c)^2+S_b(c-a)^2+S_c(a-b)^2 \quad (\star)$$

where S_a, S_b, S_c are given by

$$S_a=2b+c+\frac{1}{2}(a+b+c),\ S_b=2c+a+\frac{1}{2}(a+b+c),\ S_c=2a+b+\frac{1}{2}(a+b+c).$$

Notice that we will prove $(\star)$ for all non-negative real numbers a, b, c (we have already dismissed the condition $a+b+c=3$). Because all S_a, S_b, S_c are linear functions of a, b, c, if we replace a, b, c with $a-t, b-t, c-t$ ($t\le\min(a,b,c)$) then the differences $a-b, b-c, c-a$ are unchanged, the left hand expression of $(\star)$ is unchanged but the right hand expression of $(\star)$ is decreased. So it suffices to prove the inequality in the case $\min(a,b,c)=0$ (as we let $t=\min(a,b,c)$) and it becomes

$$a^2b\le 4,$$

which is clearly**AM-GM** inequality because $2a+b=3$.

Second Solution. WLOG, assume that b is the second greatest number of the set $\{a,b,c\}$. We certainly have

$$c(b-a)(b-c)\le 0\ \Leftrightarrow\ c(b^2-bc-ba+ac)\le 0\ \Leftrightarrow\ b^2c+c^2a\le bc(a+c)$$

and it remains to prove that

$$bc(a+c)+a^2b+abc\le 4\ \Leftrightarrow\ b(a+c)^2\le 4$$

which is also **AM-GM** inequality. Equality holds for $a=2, b=1, c=0$ up to permutation.

∇

Problem 60. *Suppose that n is an integer greater than 2. Let $a_1, a_2, ..., a_n$ be positive real numbers such that $a_1a_2...a_n=1$. Prove the following inequality*

$$\frac{a_1+3}{(a_1+1)^2}+\frac{a_2+3}{(a_2+1)^2}+...+\frac{a_n+3}{(a_n+1)^2}\ge 3.$$

(United of Kingdom TST 2005)

SOLUTION. Notice first that it is sufficient to prove the inequality in the case $n = 3$. For a bigger value of n ($n \geq 4$), we only need to choose from the set $\{a_1, a_2, ..., a_n\}$ the three smallest numbers, say a_1, a_2, a_3. Since $a_1a_2a_3 \leq 1$, there exists a positive number k such that $\frac{a}{a_1} = \frac{b}{a_2} = \frac{c}{a_3} = k \geq 1$, then

$$\sum_{i=1}^{n} \frac{a_i + 3}{(a_i + 1)^2} \geq \frac{a_1 + 3}{(a_1 + 1)^2} + \frac{a_2 + 3}{(a_2 + 1)^2} + \frac{a_3 + 3}{(a_3 + 1)^2} \geq \sum_{cyc} \frac{a + 3}{(a + 1)^2} \geq 3.$$

We will now prove that if a, b, c are positive real numbers and $abc = 1$ then

$$\frac{a + 3}{(a + 1)^2} + \frac{b + 3}{(b + 1)^2} + \frac{c + 3}{(c + 1)^2} \geq 3.$$

Let $a_1 = \frac{2}{1 + a}, b_1 = \frac{2}{1 + b}, c_1 = \frac{2}{1 + c}$. The inequality becomes

$$a_1 + b_1 + c_1 + a_1^2 + b_1^2 + c_1^2 \geq 6.$$

Since $abc = 1$, we have

$$\prod_{cyc} \left(\frac{1}{a_1} - \frac{1}{2}\right) = \frac{abc}{8} = \frac{1}{8} \Rightarrow \prod_{cyc} (2 - a_1) = a_1b_1c_1.$$

Let $x = a_1 - 1, y = b_1 - 1, z = c_1 - 1$, then $x, y, z \in [-1, 1]$ and we infer that

$$(x + 1)(y + 1)(z + 1) = (1 - x)(1 - y)(1 - z) \Rightarrow x + y + z + xyz = 0.$$

By **AM-GM** inequality, we deduce that $x^2 + y^2 + z^2 \geq 3(xyz)^{2/3} \geq 3xyz$, thus

$$a_1 + b_1 + c_1 + a_1^2 + b_1^2 + c_1^2 - 6 = \sum_{cyc} (a_1 - 1)(a_1 + 2) = \sum_{cyc} x(x + 3) \geq 0.$$

This ends the proof. Equality holds for $a = b = c = 1$.

∇

Problem 61. *Let a, b, c be non-negative real numbers with sum 2. Prove that*

$$\sqrt{a + b - 2ab} + \sqrt{b + c - 2bc} + \sqrt{c + a - 2ca} \geq 2.$$

(Pham Kim Hung)

SOLUTION. WLOG, we may assume that $a \geq b \geq c$. Let $x = a + b - 2ab, y = b + c - 2bc$ and $z = c + a - 2ca$. The inequality is equivalent to (after squaring)

$$2\sum_{cyc} \sqrt{xy} \geq 2\sum_{cyc} ab.$$

Notice that $2x = c(a+b)+(a-b)^2$ and $2y = a(b+c)+(b-c)^2$, so **Cauchy-Schwarz** inequality gives us that

$$2\sqrt{xy} \geq \sqrt{ca(a+b)(b+c)} + |(a-b)(b-c)|.$$

Applying **Cauchy-Schwarz** inequality again, we have

$$\sqrt{ca(a+b)(b+c)} = \sqrt{ca}\cdot\sqrt{(a+b)(c+b)} \geq \sqrt{ca}\left(\sqrt{ca}+b\right) = ca + b\sqrt{ca}.$$

It remains to prove that

$$\sum_{cyc}|(a-b)(b-c)| + \sum_{cyc} b\sqrt{ca} \geq \sum_{cyc} ca$$

$$\Leftrightarrow\ 2(a-c)^2 + 2(a-b)(b-c) \geq \sum_{cyc} b\left(\sqrt{c}-\sqrt{a}\right)^2.$$

Denote $\sqrt{c} = m$, $\sqrt{a}-\sqrt{b} = \alpha \geq 0$, $\sqrt{b}-\sqrt{c} = \beta \geq 0$. The inequality above becomes

$$2(\alpha+\beta+2m)^2(\alpha+\beta)^2 + 2\alpha\beta(2m+\beta)(2m+2\beta+\alpha) \geq$$

$$\geq (m+\beta)^2(\alpha+\beta)^2 + m^2\alpha^2 + (m+\alpha+\beta)^2\beta^2.$$

This last one can be reduced to $Mm^2 + Nm + P \geq 0$ where

$$M = 8(\alpha+\beta)^2 + 8\alpha\beta(\alpha+\beta)^2 + \alpha^2 + \beta^2 \geq 0.$$

$$N = 8(\alpha+\beta)^3 + 4\alpha\beta(\alpha+3\beta) - 2\beta(\alpha+\beta)^2 - 2(\alpha+\beta)\beta^2 \geq 0.$$

$$P = 2(\alpha+\beta)^4 + 2\alpha\beta^2(2\beta+\alpha) - 2\beta^2(\alpha+\beta)^2 \geq 0.$$

We are done. The equality holds for $a = b = c = \dfrac{2}{3}$ and $a = b = 1, c = 0$ up to permutation.

∇

Problem 62. *Let a, b, c be non-negative real numbers such that $a^2+b^2+c^2 = 3$. Prove that*

$$\frac{a}{b+2} + \frac{b}{c+2} + \frac{c}{a+2} \leq 1.$$

(Vasile Cirtoaje)

SOLUTION. After expanding, the inequality gets a simpler form

$$ab^2 + bc^2 + ca^2 \leq 2 + abc.$$

WLOG, suppose that b is the second greatest number in the set $\{a, b, c\}$, then

$$a(b-a)(b-c) \le 0 \Leftrightarrow a^2b + abc \ge ab^2 + ca^2.$$

Therefore suffices to prove that

$$2 \ge a^2b + bc^2 \Leftrightarrow b(a^2+c^2) \le 2 \Leftrightarrow b(3-b^2) \le 2 \Leftrightarrow (b-1)^2(b+2) \ge 0,$$

which is obvious. Equality holds for $a = b = c = 1$ or $a = 0, b = 1, c = \sqrt{2}$ up to permutation.

∇

Problem 63. *Let a, b, c be non-negative real numbers. Prove that*

$$\frac{a(b+c)}{a^2+bc} + \frac{b(c+a)}{b^2+ca} + \frac{c(a+b)}{c^2+ab} \ge 2.$$

(Pham Kim Hung)

SOLUTION. The inequality is equivalent to

$$\sum_{cyc} a(b+c)(b^2+ca)(c^2+ab) \ge 2(a^2+bc)(b^2+ca)(c^2+ab)$$

$$\Leftrightarrow \sum_{cyc} a^4(b^2+c^2) + 3abc\sum_{cyc} a^2(b+c) \ge 4a^2b^2c^2 + 2\sum_{cyc} a^3b^3 + 2abc\sum_{cyc} a^3 \ (\star)$$

According to the identity

$$(a-b)^2(b-c)^2(c-a)^2$$

$$= \sum_{cyc} a^4(b^2+c^2) + 2abc\sum_{cyc} a^2(b+c) - 2\sum_{cyc} a^3b^3 - 6a^2b^2c^2 - 2abc\sum_{cyc} a^3.$$

we can rewrite $(\star)$ as

$$(a-b)^2(b-c)^2(c-a)^2 + 2a^2b^2c^2 + abc\sum_{sym} a^2(b+c) \ge 0.$$

which is obvious. Equality holds for $a = b, c = 0$ or permutations.

Comment. According to the same identity, we can prove the following inequality (notice that without this identity, these problems are really hard)

★ *Let a, b, c be three non-negative real numbers. Prove that*

$$\frac{a(b+c-a)}{a^2+2bc} + \frac{b(c+a-b)}{b^2+2ca} + \frac{c(a+b-c)}{c^2+2ab} \ge 0.$$

∇

Problem 64. *Suppose that a, b, c are the side lengths of a triangle. Prove that*

$$\frac{a}{b+c}+\frac{b}{a+c}+\frac{c}{a+b}+\frac{ab+bc+ca}{a^2+b^2+c^2} \le \frac{5}{2}.$$

SOLUTION. With the following identities

$$-3+\sum_{cyc}\frac{2a}{b+c}=\sum_{cyc}\frac{(a-b)^2}{(a+c)(b+c)},$$

$$2-\frac{2(ab+bc+ca)}{a^2+b^2+c^2}=\frac{(a-b)^2+(b-c)^2+(c-a)^2}{a^2+b^2+c^2},$$

we can transform our inequality to $S_a(b-c)^2+S_b(a-c)^2+S_c(a-b)^2 \ge 0$ where

$$S_a=1-\frac{a^2+b^2+c^2}{(a+b)(a+c)}, S_b=1-\frac{a^2+b^2+c^2}{(b+a)(b+c)}, S_c=1-\frac{a^2+b^2+c^2}{(c+a)(c+b)}.$$

WLOG, suppose that $a \ge b \ge c$. Then, clearly, $S_a \ge 0$. Since a, b, c are the side lengths of a triangle, we get that $a \le b+c$ and $\frac{a-c}{a-b} \ge \frac{b}{c} \ge \frac{a+b}{a+c}$. Moreover

$$S_b=\frac{a(b+c-a)+c(b-c)}{(a+b)(b+c)} \ge \frac{c(b-c)}{(a+b)(b+c)},$$
$$S_c=\frac{a(b+c-a)+b(c-b)}{(a+c)(c+b)} \ge \frac{b(c-b)}{(a+c)(c+b)},$$

so we can conclude that

$$\sum_{cyc} S_a(b-c)^2 \ge (a-b)^2\left(\frac{b^2}{c^2}\cdot S_b+S_c\right)$$

$$\ge \frac{(a-b)^2}{c^2}\left(\frac{b^2c(b-c)}{(a+b)(b+c)}+\frac{c^2b(c-b)}{(a+c)(c+b)}\right)$$

$$=\frac{(a-b)^2(b-c)b}{(a+b)(b+c)}\left(\frac{b}{c}-\frac{a+b}{a+c}\right) \ge 0.$$

Equality holds for $a=b=c$ or $a=b, c=0$ or permutations.

∇

Problem 65. *Let a, b, c be non-negative real numbers. Prove that*

$$\frac{1}{\sqrt{a^2+bc}}+\frac{1}{\sqrt{b^2+ca}}+\frac{1}{\sqrt{c^2+ab}} \ge \frac{6}{a+b+c}.$$

(Pham Kim Hung)

SOLUTION. **First solution.** Taking into account problem 15, we have

$$\frac{1}{\sqrt{a^2+bc}}+\frac{1}{\sqrt{b^2+ca}}+\frac{1}{\sqrt{c^2+ab}} \geq \frac{9}{\sqrt{a^2+bc}+\sqrt{b^2+ca}+\sqrt{c^2+ab}} \geq \frac{6}{a+b+c}.$$

Second solution. Applying **AM-GM** inequality, directly we have

$$\frac{1}{\sqrt{a^2+bc}}+\frac{1}{\sqrt{b^2+ca}}+\frac{1}{\sqrt{c^2+ab}} \geq \frac{3}{\sqrt[6]{(a^2+bc)(b^2+ca)(c^2+ab)}}.$$

It remains to prove that if $a+b+c=2$ then $(a^2+bc)(b^2+ca)(c^2+ab) \leq 1$. WLOG, we may assume that $a \geq b \geq c$, then

$$\left(a+\frac{c}{2}\right)^2 \geq a^2+bc\,;$$

$$\left(b^2+c^2+ab+ac\right)^2 \geq 4(b^2+ca)(c^2+ab)\,;$$

Moreover,

$$4\left(a+\frac{c}{2}\right)(b^2+c^2+ab+ac)-(a+b+c)^3$$
$$=-(a-b)^2(a+b)+(ac^2-3a^2c)+(c^3-bc^2-b^2c) \leq 0.$$

We conclude that

$$(a^2+bc)(b^2+ca)(c^2+ab) \leq \frac{1}{4}\left(a+\frac{c}{2}\right)^2\left(b^2+c^2+ab+ac\right)^2 \leq \frac{1}{64}(a+b+c)^6=1.$$

This ends the proof. Equality holds for $a=b, c=0$ or permutation.

∇

Problem 66. *Let a,b,c be positive real numbers. Prove that*

$$\frac{a^3}{2a^2-ab+2b^2}+\frac{b^3}{2b^2-bc+2c^2}+\frac{c^3}{2c^2-ca+2a^2} \geq \frac{a+b+c}{3}.$$

(Nguyen Viet Anh)

SOLUTION. Rewrite the inequality form as follows

$$\sum_{cyc}\frac{a^3}{2a^2-ab+2b^2}-\frac{1}{3}\sum_{cyc}a=\sum_{cyc}\frac{a(a^2+ab-2b^2)}{3(2a^2-ab+2b^2)}$$
$$=\sum_{cyc}(a-b)\left(\frac{a(2a+b)}{3(2a^2-ab+2b^2)}-\frac{1}{3}\right)$$
$$=\frac{1}{3}\sum_{cyc}\frac{(a-b)^2(2b-a)}{2a^2-ab+2b^2}.$$

We may assume that $a = \max\{a, b, c\}$. If $\left\{\frac{a}{b}, \frac{b}{c}, \frac{c}{a}\right\} \in (0, 2]$, we are done. Otherwise, we need to consider some subcases

(i). The first case. $a \geq b \geq c$. If $a \geq 2b$ then

$$\frac{a^3}{2a^2 - ab + 2b^2} \geq \frac{a}{2}\,; \quad \frac{b^3}{2b^2 - bc + 2c^2} \geq \frac{b}{3}\,;$$

$$\Rightarrow \frac{a^3}{2a^2 - ab + 2b^2} + \frac{b^3}{2b^2 - bc + 2c^2} + \frac{c^3}{2c^2 - ca + 2a^2} \geq \frac{a}{2} + \frac{b}{3} \geq \frac{a+b+c}{3}.$$

Otherwise, $b \geq 2c$. Then

$$\frac{a^3}{2a^2 - ab + 2b^2} + \frac{b^3}{2b^2 - bc + 2c^2} + \frac{c^3}{2c^2 - ca + 2a^2} \geq \frac{a}{3} + \frac{b}{2} \geq \frac{a+b+c}{3}.$$

(ii). The second case. If $a \geq c \geq b$, then $0 \leq \frac{b}{c} \leq 1, 0 \leq \frac{c}{a} \leq 1$. We may assume that $a \geq 2b$, then

$$\frac{a^3}{2a^2 - ab + 2b^2} \geq \frac{a}{2} \tag{1}$$

We will prove now that

$$\frac{b^3}{2b^2 - bc + 2c^2} \geq \frac{b}{3} - \frac{c}{9} \tag{2}$$

Indeed, this inequality is equivalent to

$$f(c) = 2c^3 - 7c^2b + 5cb^2 + 3b^3 \geq 0.$$

With the condition $c \geq b$, $f'(c) = 6c^2 - 14cb + 5b^2$ has exactly one root $c_0 = \frac{(7+\sqrt{19})\,b}{16}$, therefore $f(c) \geq f(c_0) \geq 0$. (2) is proved.

Similarly, we will prove that

$$\frac{a}{6} + \frac{c^3}{2c^2 - ca + 2a^2} \geq \frac{4c}{9} \tag{3}$$

Indeed, this inequality is equivalent to

$$6a^3 - 19a^2c + 14ac^2 + 2c^3 \geq 0,$$

which is true due **AM-GM** inequality. Adding inequalities (1), (2) and (3), we get the desired result. Equality holds for $a = b = c$.

∇

Problem 67. *Prove that for all positive real numbers a, b, c*

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \geq 3\sqrt[4]{\frac{a^4 + b^4 + c^4}{3}}.$$

SOLUTION. Applying **Hölder** inequality, we obtain

$$\left(\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\right)\left(\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\right)(a^2b^2+b^2c^2+c^2a^2) \ge (a^2+b^2+c^2)^3.$$

Let $x=a^2, y=b^2, z=c^2$. It remains to prove that

$$(x+y+z)^3 \ge 3(xy+yz+zx)\sqrt{3(x^2+y^2+z^2)}.$$

$$\Leftrightarrow \frac{(x+y+z)^2}{xy+yz+zx} \ge \frac{3\sqrt{3(x^2+y^2+z^2)}}{x+y+z}$$

$$\Leftrightarrow \frac{(x-y)^2+(y-z)^2+(z-x)^2}{2(xy+yz+zx)} \ge \frac{3\left((x-y)^2+(y-z)^2+(z-x)^2\right)}{(x+y+z)\left(x+y+z+\sqrt{3(x^2+y^2+z^2)}\right)}$$

$$\Leftrightarrow 6(xy+yz+zx) \le (x+y+z)\left(x+y+z+\sqrt{3(x^2+y^2+z^2)}\right)$$

which is obvious. Equality holds for $x=y=z$ or equivalently $a=b=c$.

Comment. By a similar approach, we can prove the following inequality

★ *Let a,b,c be positive real numbers. Prove that*

$$\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b} \ge \frac{3}{2}\cdot\sqrt[4]{\frac{a^4+b^4+c^4}{3}}.$$

Moreover, an extended result for four numbers is also true

★ *Let a,b,c,d be positive real numbers. Prove that*

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{d}+\frac{d^2}{a} \ge 2\sqrt{2}\sqrt[4]{a^4+b^4+c^4+d^4},$$

∇

Problem 68. *Let a,b,c be non-negative real numbers with sum 3. Prove that*

$$(a+b^2)(b+c^2)(c+a^2) \le 13+abc.$$

(Pham Kim Hung)

SOLUTION. We will first prove that if $a \ge b \ge c$ then

$$(a+b^2)(b+c^2)(c+a^2) \ge (a^2+b)(b^2+c)(c^2+a).$$

Indeed, just notice that

$$\sum_{cyc} a^3b - \sum_{cyc} ab^3 = (a+b+c)(a-b)(b-c)(a-c),$$

$$\sum_{cyc} a^2b^3 - \sum_{cyc} a^3b^2 = (ab+bc+ca)(a-b)(b-c)(c-a).$$

Therefore

$$\prod_{cyc}(a+b^2) - \prod_{cyc}(a^2+b) = (a-b)(b-c)(a-c)\left(\sum_{cyc} a - \sum_{cyc} ab\right) \geq 0$$

because

$$\sum_{cyc} a - \sum_{cyc} ab = \frac{1}{3}\left((a+b+c)^2 - 3(ab+bc+ca)\right) = \frac{1}{3}\sum_{cyc}(a-b)^2 \geq 0.$$

According to this result, we see that it's enough to consider the case $a \geq b \geq c$. Let

$$f(a,b,c) = (a+b^2)(b+c^2)(c+a^2) - abc = \sum_{cyc} a^3b + \sum_{cyc} a^2b^3 + a^2b^2c^2.$$

We will prove that $f(a,b,c) \leq f(a+c,b,0)$. Indeed

$$f(a,b,c) - f(a+c,b,0) = \sum_{cyc} a^3b + \sum_{cyc} a^2b^3 + a^2b^2c^2 - (a+c)^3b - (a+c)^2b^3$$

$$= b^3c + c^3a + b^2c^3 + c^2a^3 + a^2b^2c^2 - 3a^2bc - 3ac^2b - 2acb^3 - c^2b^3.$$

Since $a \geq b \geq c$, we get that $b^3c \leq acb^3, c^3a \leq ac^2b, b^2c^3 \leq c^2b^3$. Finally

$$c^2a^3 + a^2b^2c^2 \leq 3a^2bc$$

is true because

$$3a^2bc - c^2a^3 - a^2b^2c^2 \geq bca^2(3-a-bc) = bca^2(b+c-bc) \geq 0.$$

This inequalities imply $f(a,b,c) \leq f(a+c,b,0) = (a+c)^2b(a+c+b^2)$. It remains to prove that if $x, y \geq 0$ and $x+y=3$ $(x=a+c, y=b)$ then $x^2y(x+y^2) \leq 13$.

Indeed, the left-hand expression, changed to a function of x, becomes

$$f(x) = (9+x^2-5x)(3x^2-x^3).$$

Applying **AM-GM** inequality, we deduce that

$$f(x) \leq \frac{1}{4}(-x^3+4x^2-5x+9)^2,$$

and according to **AM-GM** inequality again, we get

$$-x^3+4x^2-5x+9 = (x-1)^2(2-x)+7 \leq 7+\frac{4}{27}.$$

Thus, we conclude that

$$f(x) \le \frac{1}{4}\left(7+\frac{4}{27}\right)^2 < 13.$$

Comment. With the same approach, we can prove the following stronger results

★ *Let a, b, c be non-negative real numbers with sum 3. Prove that*

$$(a+b^2)(b+c^2)(c+a^2) \le 13 + abc(1-2abc).$$

★ *Let a, b, c be non-negative real numbers with sum 3. Prove that*

$$(a+b^2)(b+c^2)(c+a^2) \le 13.$$

∇

Problem 69. *Let a, b, c be positive real numbers. Prove that*

$$\frac{(a+b)^2}{c^2+ab} + \frac{(b+c)^2}{a^2+bc} + \frac{(c+a)^2}{b^2+ac} \ge 6.$$

(Peter Scholze, Darij Grinberg)

SOLUTION. We have $(a+b)^2 - 2(c^2+ab) = (a^2-c^2) + (b^2-c^2)$, therefore

$$\frac{(a+b)^2}{c^2+ab} + \frac{(b+c)^2}{a^2+bc} + \frac{(c+a)^2}{b^2+ca} - 6 = \sum_{cyc} \frac{(a^2-b^2)+(a^2-c^2)}{a^2+bc}$$

$$= \sum_{cyc}(a^2-b^2)\left(\frac{1}{a^2+bc} - \frac{1}{b^2+ac}\right) = \sum_{cyc} \frac{(a-b)^2 S_c}{M}$$

where $M = (a^2+bc)(b^2+ca)(c^2+ab)$ and S_a, S_b, S_c are determined from

$$S_a = (b+c)(b+c-a)(a^2+bc),$$
$$S_b = (c+a)(c+a-b)(b^2+ac),$$
$$S_c = (a+b)(a+b-c)(c^2+ab).$$

Now suppose that $a \ge b \ge c$. Certainly, $S_c \ge 0$ and $\dfrac{(a-c)^2}{(b-c)^2} \ge \dfrac{a^2}{b^2}$, so we have

$$\sum_{cyc} S_a(b-c)^2 \ge (a-c)^2 S_b + (b-c)^2 S_a$$

$$= (c-b)^2\left(\frac{(a-c)^2}{(c-b)^2}S_b + S_a\right) \ge \frac{(c-b)^2(a^2S_b+b^2S_a)}{b^2}.$$

On the other hand

$$a^2S_b + b^2S_a = a^2(a+c)(a+c-b)(b^2+ac) + b^2(b+c)(b+c-a)(a^2+bc)$$
$$\geq a(a-b)\left(a^2(b^2+ac) - b^2(b^2+ac)\right) \geq 0.$$

We are done. Equality occurs if $a=b=c$ or $a=b, c=0$ or permutations.

∇

Problem 70. *Find the maximum value of $k=k(n)$ for which the following inequality is true for all real numbers $x_1, x_2, ..., x_n$*

$$x_1^2 + (x_1+x_2)^2 + ... + (x_1+x_2+...+x_n)^2 \geq k(x_1+x_2+...+x_n)^2.$$

(Le Hong Quy)

SOLUTION. Let $a_1, a_2, ..., a_n$ be positive real numbers, then

$$a_1y_1^2 + \frac{1}{a_1}\cdot y_2^2 + 2y_1y_2 \geq 0$$

$$a_2y_2^2 + \frac{1}{a_2}\cdot y_3^2 + 2y_2y_3 \geq 0$$

$$\ldots \quad \ldots$$

$$a_{n-1}y_{n-1}^2 + \frac{1}{a_n}\cdot y_n^2 + 2y_{n-1}y_n \geq 0.$$

Adding up these results, we obtain

$$a_1y_1^2 + \left(\frac{1}{a_1} + a_2\right)y_2^2 + ... + \left(\frac{1}{a_{n-2}} + a_{n-1}\right)y_{n-1}^2 + \frac{1}{a_{n-1}}\cdot y_n^2 + 2\sum_{i=1}^{n-1} y_iy_{i+1} \geq 0 \ (\star)$$

We will choose n numbers $a_1, a_2, ..., a_n$ such that

$$a_1 = \frac{1}{a_1} + a_2 = ... = \frac{1}{a_{n-1}} - 1.$$

With some calculations, we find out $a_k = \dfrac{\sin(k+1)\alpha}{\sin(k\alpha)}$ where $\alpha = \dfrac{2\pi}{2n+1}$. Inserting these in $(\star)$, we conclude

$$2\cos\alpha\left(\sum_{k=1}^{n} y_k^2\right) + 2\left(\sum_{k=1}^{n-1} y_ky_{k+1}\right) + y_n^2 \geq 0$$

$$\Leftrightarrow\ 2(1+\cos\alpha)\left(\sum_{k=1}^{n} y_k^2\right) \geq y_1^2 + \sum_{k=1}^{n-1}(y_k - y_{k+1})^2.$$

Setting $y_k = x_1 + x_2 + ... + x_k$ $\forall k \in \{1, 2, ..., n\}$, we obtain

$$4\cos^2\frac{\alpha}{2}\left(x_1^2 + (x_1 + x_2)^2 + ... + (x_1 + x_2 + ... + x_n)^2\right) \geq x_1^2 + x_2^2 + ... + x_n^2.$$

The best (greatest) value of k is $\dfrac{1}{4\cos^2\frac{\pi}{2n+1}}$ with equality for

$$x_k = (-1)^k\left(\sin\frac{2k\pi}{2n+1} + \sin\frac{2(k-1)\pi}{2n+1}\right).$$

Comment. In example 6.2.4, we proved (by **Cauchy-Schwarz** inequality)

$$x_1^2 + (x_1 + x_2)^2 + ... + (x_1 + x_2 + ... + x_n)^2 \leq \frac{1}{4\sin^2\frac{\pi}{2(2n+1)}}(x_1^2 + x_2^2 + ... + x_n^2).$$

What a strange and interesting coincidence! One problem asks for the maximum value, the other problem asks for the minimum value, one problem is based on the **AM-GM** inequality and the other on **Cauchy-Schwarz** inequality, but both come to two similar results, with the appearance of $\dfrac{2\pi}{2n+1}$.

∇

Problem 71. *Let $a_1, a_2, ..., a_n$ be positive real numbers with sum n. Prove that*

$$\frac{1}{a_1} + \frac{1}{a_2} + ... + \frac{1}{a_n} - n \geq \frac{8(n-1)(1 - a_1a_2...a_n)}{n^2}.$$

(Pham Kim Hung)

SOLUTION. We prove this inequality by induction. If $n = 2$, the problem is obvious

$$\frac{1}{a_1} + \frac{1}{a_2} - 2 \geq 2(1 - a_1a_2) \Leftrightarrow (1 - a_1a_2)^2 \geq 0.$$

Let's consider the problem for $n+1$ numbers with the supposition that it is true for n numbers. We assume that $a_1 \leq a_2 \leq ... \leq a_n \leq a_{n+1}$. For each $i \in \{1, 2, ..., n\}$, we denote $b_i = \dfrac{a_i}{t}$, where $t = \dfrac{a_1 + a_2 + ... + a_n}{n} \leq 1$. Applying the inductive hypothesis for $b_1, b_2, ..., b_n$, we obtain

$$\frac{1}{b_1} + \frac{1}{b_2} + ... + \frac{1}{b_n} - n \geq c(1 - b_1b_2...b_n),$$

for all $c \leq \dfrac{8(n-1)}{n^2}$. Replacing b_i with $\dfrac{a_i}{t}$, we deduce that

$$-n + \sum_{i=1}^{n}\frac{t}{a_i} \geq c\left(1 - \frac{1}{t^n}\prod_{i=1}^{n}a_i\right) \Leftrightarrow \sum_{i=1}^{n}\frac{1}{a_i} + \frac{c}{t^{n+1}}\left(\prod_{i=1}^{n}a_i\right) \geq \frac{n}{t} + \frac{c}{t} \ (\star)$$

For $n+1$ numbers, we need to prove that, if $k = \dfrac{8n}{(n+1)^2}$, then

$$-n-1+\sum_{i=1}^{n+1}\frac{1}{a_i} \geq k\left(1-\prod_{i=1}^{n+1}a_i\right) \Leftrightarrow \sum_{i=1}^{n}\frac{1}{a_i}+(ka_{n+1})\prod_{i=1}^{n}a_i+\frac{1}{a_{n+1}} \geq n+1+k$$

Let $c' = (ka_{n+1})t^{n+1}$. According to **AM-GM** inequality, we deduce that

$$a_{n+1}t^n \leq \left(\frac{a_{n+1}+nt}{n+1}\right)^{n+1} = 1,$$

hence $c' \leq kt \leq k = \dfrac{8n}{(n+1)^2} \leq \dfrac{8(n-1)}{n^2}$. On the other hand, notice that $(\star)$ holds for all $c \leq \dfrac{8(n-1)}{n^2}$. It also holds for $c = c'$, so we have

$$\sum_{i=1}^{n}\frac{1}{a_i}+(ka_{n+1})\left(\prod_{i=1}^{n}a_i\right) \geq \frac{n}{t}+ka_{n+1}t^n.$$

It remains to prove that

$$\frac{n}{t}+ka_{n+1}t^n+\frac{1}{a_{n+1}} \geq n+1+k.$$

Replacing a_{n+1} with $n+1-nt$, we obtain an equivalent inequality

$$\frac{n}{t}+\frac{1}{n+1-nt}-(n+1) \geq k\left(nt^{n+1}-(n+1)t^n+1\right)$$

$$\Leftrightarrow \quad \frac{n(n+1)}{t(n+1-nt)} \geq \frac{8n}{(n+1)^2}(1+2t+\ldots+nt^{n-1})$$

which is obvious because $t \leq 1$ and $t(n+1-nt) \leq \dfrac{(n+1)^2}{4n}$. We are done.

∇

Problem 72. *Let x, y, z be non-negative real numbers with sum 1. Prove that*

$$\sqrt{x+y^2}+\sqrt{y+z^2}+\sqrt{z+x^2} \geq 2.$$

(Phan Thanh Nam)

SOLUTION. Notice that if a, b, c, d are non-negative real numbers such that $a+b = c+d$ and $|a-b| \leq |c-d|$, then we have

$$\sqrt{a}+\sqrt{b} \geq \sqrt{c}+\sqrt{d} \ (\star)$$

Indeed, since $(a+b)^2-(a-b)^2 \geq (c+d)^2-(c-d)^2$, we have $ab \geq cd$, therefore

$$a+b+2\sqrt{ab} \geq c+d+2\sqrt{cd} \Rightarrow \sqrt{a}+\sqrt{b} \geq \sqrt{c}+\sqrt{d}.$$

According to ($\star$), we deduce that

$$\sqrt{x+y^2}+\sqrt{y+z^2} \geq (x+y)+\sqrt{z+y^2}.$$

We conclude that

$$\begin{aligned}\sqrt{x+y^2}+\sqrt{y+z^2}+\sqrt{z+x^2} &\geq (x+y)+\sqrt{z+y^2}+\sqrt{z+x^2}\\ &\geq x+y+\sqrt{(\sqrt{z}+\sqrt{z})^2+(x+y)^2}\\ &= 1-z+\sqrt{4z+(1-z)^2}=2.\end{aligned}$$

Equality holds for $x=y=z=\frac{1}{3}$ or $x=1, y=z=0$ or permutations.

∇

Problem 73. *Let a, b, c be positive real numbers with sum 3. Prove that*

$$\frac{1}{2+a^2b^2}+\frac{1}{2+b^2c^2}+\frac{1}{2+c^2a^2} \geq 1.$$

(Pham Kim Hung)

SOLUTION. According to **AM-GM** inequality, we have

$$\frac{1}{2+a^2b^2}=\frac{1}{2}-\frac{a^2b^2}{2(2+a^2b^2)} \geq \frac{1}{2}-\frac{a^2b^2}{6\sqrt[3]{a^2b^2}}=\frac{1}{2}-\frac{a^{4/3}b^{4/3}}{6}.$$

We deduce that

$$\sum_{cyc}\frac{1}{2+a^2b^2} \geq \frac{3}{2}-\frac{1}{6}\sum_{cyc}a^{4/3}b^{4/3}$$

and it remains to prove that $\sum_{cyc} a^{4/3}b^{4/3} \leq 3$. By **AM-GM** inequality again, we have

$$3\sum_{cyc}a^{4/3}b^{4/3}=3\sum_{cyc}ab\sqrt[3]{ab} \leq \sum_{cyc}ab(a+b+1)=4(ab+bc+ca)-3abc.$$

Recalling a familiar result $(a+b-c)(b+c-a)(c+a-b) \leq abc$, we have

$$(3-2a)(3-2b)(3-2c) \leq abc \Leftrightarrow 4(ab+bc+ca)-3abc \leq 9.$$

We are done. The equality holds for $a=b=c=1$.

Comment. The following general result is proposed by Gabriel Dospinescu and Vasile Cirtoaje

★ *Suppose that a, b, c are three non-negative real numbers adding up to 3. Find the maximum value of k for which the following inequality is true*

$$(ab)^k+(bc)^k+(ca)^k \leq 3.$$

Let's examine this problem. It is obviously wrong if $k \le 0$. It is obviously true if $0 < k \le 1$. Consider now the case $k \ge 2$. With the supposition $a \ge b \ge c$, we have

$$(ab)^k + (bc)^k + (ca)^k \le a^k(b+c)^k = a^k(3-a)^k \le \left(\frac{3}{2}\right)^{2k}.$$

If $1 \le k \le 2$, we let $t = \frac{a+b}{2}$ and $u = \frac{a-b}{2}$ then $a = t+u, b = t-u$. Denote

$$f(u) = c^k\left((t+u)^k + (t-u)^k\right) + (t^2-u^2)^k$$

then its derivative is

$$f'(u) = kc^k(t^2-u^2)^{k-1}\left(\frac{1}{(t-u)^{k-1}} - \frac{1}{(t+u)^{k-1}} - \frac{2u}{c^k}\right).$$

Applying **Lagrange** theorem for the function $g(x) = x^{1-k}$, we deduce that there exists a real number $t_0 \in [t-u, t+u]$ such that

$$\frac{1}{(t-u)^{k-1}} - \frac{1}{(t+u)^{k-1}} = \frac{2u(k-1)}{t_0^k}.$$

$t_0 \ge t-u \ge c$ and $k \le 2$, so we get $f'(u) \le 0$ by

$$\frac{1}{(t-u)^{k-1}} - \frac{1}{(t+u)^{k-1}} = \frac{2u(k-1)}{t_0^k} \le \frac{2u}{c^k}.$$

Thus $f(u) \le f(0)$. It remains to consider the case $a = b \ge 1 \ge c$. Denote

$$h(a) = 2a^k(3-2a)^k + a^{2k},$$

then we have

$$h'(a) = 2ka^{k-1}(3-2a)^{k-1}\left(3 - 4a + \frac{a^k}{(3-2a)^{k-1}}\right).$$

With the condition $0 < a < \frac{3}{2}$, the equation $h'(a) = 0$ has only solutions $a \ge \frac{3}{4}$, and

$$k\ln a - (k-1)\ln(3-2a) = \ln(4a-3).$$

We denote $q(a) = k\ln a - (k-1)\ln(3-2a) - \ln(4a-3)$, then

$$aq'(a) = k + \frac{(k-1)a}{3-a} - \frac{4a}{4a-3}.$$

Notice that both $\frac{a}{3-a}$ and $\frac{-a}{4a-3}$ are increasing functions, therefore the equation $aq'(a) = 0$ has no more than one root, therefore the function $q(a) = 0$ has no more than two roots and therefore the equation $h'(a) = 0$ has no more than two roots.

Because $h'(1) = 0$ and $q'(1) = k + 2(k-1) - 4 = 3k - 6 \le 0$, we easily deduce from the variance table that

$$h(a) \le \max\left\{h(1), h\left(\frac{3}{2}\right)\right\}.$$

We conclude that for all positive real numbers k then

$$(ab)^k + (bc)^k + (ca)^k \le \max\left\{3, \left(\frac{3}{2}\right)^{2k}\right\},$$

and equality is reached for every k. Therefore the maximum constant k that we are looking for is $\dfrac{\ln 3}{2(\ln 3 - \ln 2)}$.

∇

Problem 74. *Consider the positive real constants m, n, such that $3n^2 > m^2$. For real numbers a, b, c such that $a + b + c = m, a^2 + b^2 + c^2 = n^2$, find the maximum and minimum of*

$$P = a^2b + b^2c + c^2a.$$

(Le Trung Kien, Vo Quoc Ba Can)

SOLUTION. Let $a = x + \dfrac{m}{3}, b = y + \dfrac{m}{3}, c = z + \dfrac{m}{3}$. From the given conditions, we get that $x + y + z = 0$ and $x^2 + y^2 + z^2 = \dfrac{3n^2 - m^2}{3}$. The expression P becomes

$$P = x^2y + y^2z + z^2x + \frac{m^3}{9}.$$

Notice that

$$\begin{aligned}
&\sum_{cyc}\left(3x\sqrt{\frac{2}{3n^2 - m^2}} - \frac{18xy}{3n^2 - m^2} - 1\right)^2 \\
&= 3 + \frac{18}{3n^2 - m^2}\left(\sum_{cyc} x\right)^2 + \frac{324}{(3n^2 - m^2)^2}\sum_{cyc} x^2y^2 \\
&\quad - 6\sqrt{\frac{2}{3n^2 - m^2}}\sum_{cyc} x - 54\left(\frac{2}{3n^2 - m^2}\right)^{3/2}\sum_{cyc} x^2y \\
&= 3 + \frac{324}{(3n^2 - m^2)^2}\sum_{cyc} x^2y^2 - 54\left(\frac{2}{3n^2 - m^2}\right)^{3/2}\sum_{cyc} x^2y.
\end{aligned}$$

Since $x + y + z = 0$, we get $xy + yz + zx = -\dfrac{1}{2}(x^2 + y^2 + z^2) = -\dfrac{3n^2 - m^2}{6}$. Therefore

$$\sum_{cyc} x^2y^2 = \left(\sum_{cyc} xy\right)^2 - 2xyz\sum_{cyc} x = \left(\sum_{cyc} xy\right)^2 = \frac{(3n^2 - m^2)^2}{36}$$

and we get

$$12 - 54\left(\frac{2}{3n^2 - m^2}\right)^{3/2} \sum_{cyc} x^2 y \geq 0$$

or in other words,

$$\sum_{cyc} x^2 y \leq \frac{2}{9}\left(\frac{3n^2 - m^2}{2}\right)^{3/2}.$$

If we choose

$$x = \frac{\sqrt{2(3n^2 - m^2)}}{3} \cos\frac{2\pi}{9}, y = \frac{\sqrt{2(3n^2 - m^2)}}{3} \cos\frac{4\pi}{9}, z = \frac{\sqrt{2(3n^2 - m^2)}}{3} \cos\frac{8\pi}{9},$$

then

$$P = \frac{2}{9}\left(\frac{3n^2 - m^2}{2}\right)^{3/2} + \frac{m^3}{9}$$

So

$$\max P = \frac{2}{9}\left(\frac{3n^2 - m^2}{2}\right)^{3/2} + \frac{m^3}{9}$$

Similarly, by considering the expression

$$\sum_{cyc}\left(3x\sqrt{\frac{2}{3n^2 - m^2}} + \frac{18xy}{3n^2 - m^2} + 1\right)^2,$$

we easily conclude that

$$\min P = -\frac{2}{9}\left(\frac{3n^2 - m^2}{2}\right)^{3/2} - \frac{m^3}{9}.$$

The problem is completely solved.

∇

Problem 75. *Suppose that a, b, c are three positive real numbers verifying*

$$(a + b + c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) = 13.$$

Find the minimum and maximum values of the expression

$$P = \frac{a^3 + b^3 + c^3}{abc}.$$

(Pham Kim Hung)

SOLUTION. Denote

$$x = \sum_{cyc} \frac{a}{b}\ ;\ y = \sum_{cyc} \frac{b}{a}\ ;\ m = \sum_{cyc} \frac{a^2}{bc}\ ;\ n = \sum_{cyc} \frac{bc}{a^2}\ ;$$

We have $x+y=10$ and

$$x^3 = 3(m+n)+6+\sum_{cyc}\frac{a^3}{b^3} \quad ; \quad y^3 = 3(m+n)+6+\sum_{cyc}\frac{b^3}{a^3} \ .$$

The identities above yield that

$$x^3+y^3 = \left(a^3+b^3+c^3\right)\left(\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}\right)+6(m+n)+9.$$

We also have

$$mn = 3+\sum_{cyc}\frac{a^3}{b^3}+\sum_{cyc}\frac{b^3}{a^3} = \left(a^3+b^3+c^3\right)\left(\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}\right)$$

and

$$xy = \left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right) = 3+m+n$$

thus we infer that

$$10^3-30(3+m+n) = mn+6(m+n)+9$$

$$\Leftrightarrow \ 10^3-99 = mn+36(m+n)$$

so m,n are two positive roots of the quadratic

$$f(t) = t^2-(xy-3)t+(1009-36xy).$$

Letting now $r=xy$, we can determine

$$2m = (xy-3)\pm\sqrt{(xy-3)^2-4(1009-36xy)}$$

Consider the function $g(r) = r-3-\sqrt{r^2+138r-4027}$ for $0 \le r \le 25$. Notice that

$$g'(r) = 1-\frac{2r+138}{2\sqrt{r^2+138r-4027}} < 0$$

so we conclude $11-2\sqrt{3} \le m \le 11+2\sqrt{3}$, with equality for $x-y=(a-b)(b-c)(c-a)=0$. The minimum of m is $11-2\sqrt{3}$, attained for $a=b=\left(2+\sqrt{3}\right)c$ up to permutation. The maximum of m is $11+2\sqrt{3}$, attained for $a=b=\left(2-\sqrt{3}\right)c$ up to permutation.

∇

Problem 76. *Prove that for all positive real numbers a,b,c,d,e,*

$$\frac{a+b}{2}\cdot\frac{b+c}{2}\cdot\frac{c+d}{2}\cdot\frac{d+e}{2}\cdot\frac{e+a}{2} \le \frac{a+b+c}{3}\cdot\frac{b+c+d}{3}\cdot\frac{c+d+e}{3}\cdot\frac{d+e+a}{3}\cdot\frac{e+a+b}{3}.$$

SOLUTION. We will first prove that for all $a, b > 0$ and $a + b \le 1$

$$\left(\frac{1}{a}-1\right)\left(\frac{1}{b}-1\right) \ge \left(\frac{2}{a+b}-1\right)^2 \quad (\star)$$

Indeed, this result can be rewritten in the following form

$$\frac{1}{ab}-\frac{1}{a}-\frac{1}{b} \ge \frac{4}{(a+b)^2}-\frac{4}{a+b} \Leftrightarrow \frac{1}{ab}-\frac{4}{(a+b)^2} \ge \frac{1}{a}+\frac{1}{b}-\frac{4}{a+b}$$
$$\Leftrightarrow \frac{(a-b)^2}{ab(a+b)^2} \ge \frac{(a-b)^2}{ab(a+b)} \Leftrightarrow (a-b)^2(1-a-b) \ge 0.$$

Return now to the original problem. We may assume that $a+b+c+d+e=1$, then

$$\prod_{cyc}\left(\frac{a+b}{2}\right) \le \prod_{cyc}\left(\frac{a+b+c}{3}\right) \Leftrightarrow \prod_{cyc}\left(\frac{a+b+c}{d+e}\right) \ge \frac{3^5}{2^5} \Leftrightarrow \prod_{cyc}\left(\frac{1}{a+b}-1\right) \ge \frac{3^5}{2^5}.$$

According to $(\star)$, we deduce that

$$\left(\frac{1}{d+e}-1\right)\left(\frac{1}{a+b}-1\right) \ge \left(\frac{2}{d+e+a+b}-1\right)^2 = \left(\frac{2}{1-c}-1\right)^2.$$

This result shows that

$$\prod_{cyc}\left(\frac{1}{a+b}-1\right) \ge \prod_{cyc}\left(\frac{2}{1-c}-1\right) = \prod_{cyc}\left(\frac{1+c}{1-c}\right).$$

The function $f(x) = \ln(1+x) - \ln(1-x)$ is convex because its second derivative is $f''(x) = \frac{-1}{(1+x)^2} + \frac{1}{(1-x)^2} \ge 0$, therefore **Jensen** inequality claims that

$$\sum_{cyc} f(a) \ge 5f\left(\frac{1}{5}\right) = 5\ln\left(\frac{3}{2}\right) \Rightarrow \prod_{cyc}\left(\frac{1+c}{1-c}\right) \ge \frac{3^5}{2^5}.$$

We are done and the equality holds for $a = b = c = d = e$.

Comment. By the same method, we can prove the following general result

★ *Let $a_1, a_2, ..., a_{2n+1}$ be positive real numbers. For each $k \in \{1, 2, ..., 2n+1\}$, we define numbers S_k, P_k as follow*

$$S_k = \frac{a_{k+1}+a_{k+2}+\ldots+a_{k+n}}{n} \quad ; \quad P_k = \frac{a_{k+1}+a_{k+2}+\ldots+a_{k+n+1}}{n+1} \quad ;$$

with $a_{k+2n+1} = a_k$. Prove that

$$S_1 \cdot S_2 \cdots S_{2n+1} \le P_1 \cdot P_2 \cdots P_{2n+1}.$$

∇

Problem 77. *Let a, b, c be non-negative real numbers. Prove that*

$$\frac{a^4}{a^3+b^3}+\frac{b^4}{b^3+c^3}+\frac{c^4}{c^3+a^3} \ge \frac{a+b+c}{2}.$$

SOLUTION. Notice that

$$\frac{2a^4}{a^3+b^3}-a-\frac{3(a-b)}{2}=(a-b)\left(\frac{a(a^2+ab+b^2)}{a^3+b^3}-\frac{3}{2}\right)=\frac{2b^2+ab-b^2}{3(a^3+b^3)}(a-b)^2.$$

Therefore the inequality can be transformed to

$$S_a(b-c)^2+S_b(a-c)^2+S_c(a-b)^2 \ge 0,$$

in which the coefficients S_a, S_b, S_c are

$$S_a=\frac{3c^2+bc-b^2}{b^3+c^3} \quad ; \quad S_b=\frac{3a^2+ca-c^2}{c^3+a^3} \quad ; \quad S_c=\frac{3b^2+ab-a^2}{a^3+b^3}.$$

The first case. If $a \ge b \ge c$, then clearly $S_b \ge 0$ and

$$S_b+2S_c=\frac{3a^2+ca-c^2}{c^3+a^3}+\frac{2(3b^2+ab-a^2)}{a^3+b^3} \ge \frac{3a^2}{c^3+a^3}-\frac{2a^2}{a^3+b^3} \ge 0.$$

$$a^2S_b+2b^2S_a=\frac{a^2(3a^2+ca-c^2)}{c^3+a^3}+\frac{2b^2(3c^2+bc-b^2)}{b^3+c^3} \ge \frac{3a^4}{c^3+a^3}-\frac{2b^4}{c^3+b^3} \ge 0.$$

So we conclude that

$$2\sum_{cyc} S_a(b-c)^2 \ge (S_b+2S_c)(a-b)^2+(b-c)^2\left(2S_a+\frac{a^2}{b^2}S_b\right) \ge 0.$$

The second case. If $c \ge b \ge a$, then clearly $S_a, S_c \ge 0$ and

$$S_a+2S_b=\frac{3c^2+bc-b^2}{b^3+c^3}+\frac{2(3a^2+ca-c^2)}{c^3+a^3} \ge \frac{3c^2+bc}{b^3+c^3}-\frac{2c^2}{a^3+c^3} \ge 0.$$

$$S_c+2S_b=\frac{3b^2+ab-a^2}{a^3+b^3}+\frac{2(3a^2+ca-c^2)}{c^3+a^3} \ge \frac{3b^2}{a^3+b^3}-\frac{2c^2}{c^3+a^3} \ge 0.$$

We conclude that

$$2\sum_{cyc} S_a(b-c)^2 \ge (S_a+2S_b)(b-c)^2+(2S_b+S_c)(a-b)^2 \ge 0.$$

The proof is finished and the equality holds for $a=b=c$.

∇

Problem 78. *Let a, b, c be positive real numbers. Prove that*

$$\sqrt{\frac{a^3}{a^2+ab+b^2}}+\sqrt{\frac{b^3}{b^2+bc+c^2}}+\sqrt{\frac{c^3}{c^2+ca+a^2}} \ge \frac{\sqrt{a}+\sqrt{b}+\sqrt{c}}{\sqrt{3}}.$$

(Le Trung Kien)

Solution. Let $x^2 = a, y^2 = b$ and $z^2 = c$. The inequality becomes

$$\frac{x^3}{\sqrt{x^4+x^2y^2+y^4}} + \frac{y^3}{\sqrt{y^4+y^2z^2+z^4}} + \frac{z^3}{\sqrt{z^4+z^2x^2+x^4}} \ge \frac{x+y+z}{\sqrt{3}}.$$

Squaring both sides, we obtain an equivalent form

$$\sum_{cyc} \frac{x^6}{x^4+x^2y^2+y^4} + \sum_{cyc} \frac{2x^3y^3}{\sqrt{(x^4+x^2y^2+y^4)(y^4+y^2z^2+z^4)}} \ge \frac{1}{3}\left(\sum_{cyc} x^2 + 2\sum_{cyc} xy\right).$$

Notice that $\sum_{cyc} \frac{x^6-y^6}{x^4+x^2y^2+y^4} = 0$, so the above inequality can be transformed to

$$\sum_{cyc} \frac{6x^3y^3}{\sqrt{(x^4+x^2y^2+y^4)(y^4+y^2z^2+z^4)}} \ge \frac{1}{2}\sum_{cyc}\left(x^2+y^2+4xy - \frac{3(x^6+y^6)}{x^4+x^2y^2+y^4}\right)$$

$$\Leftrightarrow \sum_{cyc} \frac{6x^3y^3}{\sqrt{(x^4+x^2y^2+y^4)(y^4+y^2z^2+z^4)}} \ge \sum_{cyc} \frac{6x^3y^3-(x-y)^4(x+y)^2}{x^4+x^2y^2+y^4}.$$

Then, the following sequences

$$\left(\frac{x^3y^3}{\sqrt{x^4+x^2y^2+y^4}}, \frac{y^3z^3}{\sqrt{y^4+y^2z^2+z^4}}, \frac{z^3x^3}{\sqrt{z^4+z^2x^2+x^4}}\right),$$

$$\left(\frac{1}{\sqrt{x^4+x^2y^2+y^4}}, \frac{1}{\sqrt{y^4+y^2z^2+z^4}}, \frac{1}{\sqrt{z^4+z^2x^2+x^4}}\right).$$

are monotone in the opposite order, so **Rearrangement** inequality shows that

$$\sum_{cyc} \frac{x^3y^3}{\sqrt{(x^4+x^2y^2+y^4)(y^4+y^2z^2+z^4)}} \ge \sum_{sym} \frac{x^3y^3}{x^4+x^2y^2+y^4}.$$

This ends the proof. Equality holds for $a=b=c$.

∇

Problem 79. *Let a, b, c be non-negative real numbers with sum 2. Prove that*

$$\frac{ab}{1+c^2} + \frac{bc}{1+a^2} + \frac{ca}{1+b^2} \le 1.$$

(Pham Kim Hung)

Solution. We denote $x = ab+bc+ca$ and $p = abc$. According to the identities

$$A = (a-b)^2(b-c)^2(c-a)^2 = 4x^2(1-x) + 4(9x-8)p - 27p^2,$$

$$B = \sum_{cyc} a^2(a-b)(a-c) = 12p + 4(1-x)(4-x),$$

we can rewrite our inequality as

$$(1-x)(5-2x+x^2)+(6x-2)p-2p^2 \geq 0$$

$$\Leftrightarrow \; 6A+\frac{5}{2}(1+9x)B+(1-x)^2(365-147x) \geq 0,$$

which is obvious because $x \leq \frac{4}{3}$. The equality holds for $a=b=c$.

∇

Problem 80. *Suppose that $a_1, a_2, ..., a_n$ are non-negative real numbers which add up to n. Find the minimum of the expression*

$$S = a_1^2+a_2^2+...+a_n^2+a_1a_2...a_n\left(\frac{1}{a_1}+\frac{1}{a_2}+...+\frac{1}{a_n}\right).$$

(Pham Kim Hung)

SOLUTION. Consider the following function

$$F = f(a_1,a_2,...,a_n) = a_1^2+a_2^2+...+a_n^2+a_1a_2...a_n\left(\frac{1}{a_1}+\frac{1}{a_2}+...+\frac{1}{a_n}\right).$$

Surprisingly, these exceptional identities will help

$$f(a_1,a_2,..,a_n)-f(0,a_1+a_2,a_3,...,a_n)$$
$$= a_1a_2\left(a_3a_4...a_n\left(\frac{1}{a_3}+\frac{1}{a_4}+...+\frac{1}{a_n}\right)-2\right)$$
$$f(a_1,a_2,...,a_n)-f\left(\frac{a_1+a_2}{2},\frac{a_1+a_2}{2},a_3,...,a_n\right)$$
$$= \frac{(a_1-a_2)^2}{4}\left(2-a_3a_4...a_n\left(\frac{1}{a_3}+\frac{1}{a_4}+...+\frac{1}{a_n}\right)\right)$$

Therefore at least one of the following inequalities must be true

$$F \geq f(0,a_1,a_2,...,a_n) \;\; (\star) \;\; ; \;\; F \geq f\left(\frac{a_1+a_2}{2},\frac{a_1+a_2}{2},a_3,...,a_n\right) \;\; (\star\star) \; .$$

WLOG, we may assume $a_1 \geq a_2 \geq ... \geq a_n$. Consider the transformation

$$(a_i,a_j) \to \left(\frac{a_i+a_j}{2},\frac{a_i+a_j}{2}\right).$$

If $\left(\prod_{k\neq i,j} a_k\right)\left(\sum_{k\neq i,j}\frac{1}{a_k}\right) < 2$ then F decreases after each of these transformations. If we have $a_{i_1}a_{i_2}...a_{i_{n-2}}\left(\frac{1}{a_{i_1}}+\frac{1}{a_{i_2}}+...+\frac{1}{a_{i_{n-2}}}\right) < 2$ after such transformations

for every $\{i_1, i_2, ..., i_{n-2}\} \subset \{1, 2, ..., n\}$, $(\star\star)$ claims that the minimum of F is only attained for n equal variables. In this case, we have $\min F = 2n$.

Otherwise, there exists a certain transformation for which

$$\left(\prod_{k=1}^{n-2} a_{i_k}\right)\left(\sum_{k=1}^{n-2} \frac{1}{a_{i_k}}\right) \geq 2.$$

According to $(\star)$, F only attains its minimum if the smallest element of the set $\{a_1, a_2, ..., a_n\}$ is equal to 0. In this case, we obtain

$$F = g(a_1, a_2, ..., a_{n-1}) = a_1^2 + a_2^2 + ... + a_{n-1}^2 + a_1 a_2 ... a_{n-1}.$$

By the same approach, we can conclude that at least one of the following inequalities will hold

$$g(a_1, a_2, ..., a_{n-1}) \geq g(0, a_1 + a_2, a_3, ..., a_{n-1})$$
$$g(a_1, a_2, ..., a_{n-1}) \geq g\left(\frac{a_1 + a_2}{2}, \frac{a_1 + a_2}{2}, a_3, ..., a_{n-1}\right),$$

and using the same reasoning for the function g, we deduce that $g(a_1, a_2, ..., a_n)$ attains its minimum if and only if all numbers of the set $\{a_1, a_2, ..., a_{n-1}\}$ are equal together or $n-2$ numbers are equal and another is equal to 0. This fact claims that

$$\min F = \min\left(2n, \frac{n^2}{n-2}, \frac{n^2}{n-1} + \left(\frac{n}{n-1}\right)^{n-1}\right).$$

Comment. The following result can be deduced as a part of the solution above

★ *Let $a_1, a_2, ..., a_n$ be non-negative real numbers with sume n. For all $k \in \mathbb{R}$,*

$$a_1^2 + a_2^2 + ... + a_n^2 + k a_1 a_2 ... a_n \geq \min\left\{n + k, \frac{n^2}{n-1}\right\}.$$

∇

Problem 81. *Let x, y, z be positive real numbers satisfying $2xyz = 3x^2 + 4y^2 + 5z^2$. Find the minimum of the expression $P = 3x + 2y + z$.*

(Pham Kim Hung)

SOLUTION. Let $a = 3x, b = 2y, z = c$ We then obtain

$$a + b + c = 3x + 2y + z\ ,\ a^2 + 3b^2 + 15c^2 = abc.$$

According to the weighted **AM-GM** inequality, we have that

$$a + b + c \geq (2a)^{1/2}(3b)^{1/3}(6c)^{1/6},$$

$$a^2+3b^2+15c^2 \geq (4a^2)^{1/4}(9b^2)^{3/9}(36c^2)^{15/36} = (4a^2)^{1/4}(9b^2)^{1/3}(36c^2)^{5/12}.$$

Multiplying the results above, we obtain

$$(a+b+c)(a^2+3b^2+15c^2) \geq 36abc \Rightarrow a+b+c \geq 36.$$

So the minimum of $3x+2y+z$ is 36, attained for $x=y=z=6$.

Comment. 1. Let's consider the following general result

★ *Let a,b,c,x,y,z be positive real numbers verifying $ax^2+by^2+cz^2=xyz$.*

a. Prove that there exists exactly one positive real number k for which

$$\frac{1}{2\sqrt{k}} = \frac{1}{\sqrt{k}+\sqrt{k+a}} + \frac{1}{\sqrt{k}+\sqrt{k+b}} + \frac{1}{\sqrt{k}+\sqrt{k+c}}.$$

b. With this value of k, prove that

$$x+y+z \geq \frac{(\sqrt{k}+\sqrt{k+a})(\sqrt{k}+\sqrt{k+b})(\sqrt{k}+\sqrt{k+c})}{\sqrt{k}}.$$

SOLUTION. Part (a) is fairly simple. Consider the following function

$$f(k) = \frac{\sqrt{k}}{\sqrt{k}+\sqrt{k+a}} + \frac{\sqrt{k}}{\sqrt{k}+\sqrt{k+b}} + \frac{\sqrt{k}}{\sqrt{k}+\sqrt{k+c}} - \frac{1}{2}.$$

Since $f(k)$ is an increasing function of k, and, $f(0) = -1/2, \lim_{k\to\infty} f(k) = 1$, the continuity of f claims that the equation $f(k)=0$ has exactly one positive root.

To prove (b), we let m,n,p,m_1,n_1,p_1 be positive real numbers such that

$$m+n+p=1, am_1+bn_1+cp_1=1.$$

By the weighted **AM-GM** inequality, we have

$$x+y+z \geq \left(\frac{x}{m}\right)^m \left(\frac{y}{n}\right)^n \left(\frac{z}{p}\right)^p,$$

$$ax^2+by^2+cz^2 \geq \left(\frac{x^2}{m_1}\right)^{am_1} \left(\frac{y^2}{bn_1}\right)^{n_1} \left(\frac{z^2}{p_1}\right)^{cp_1}.$$

The results above combine to show that

$$(x+y+z)(ax^2+by^2+cz^2) \geq \frac{x^{m+2am_1}y^{n+2bn_1}z^{p+2cp_1}}{m^m n^n p^p m_1^{am_1} n_1^{bn_1} p_1^{cp_1}}.$$

We will choose six numbers m,n,p,m_1,n_1,p_1 verifying the following conditions

- $m+2am_1 = n+2bn_1 = p+2cp_1 = 1.$

- $\dfrac{x}{m} = \dfrac{y}{n} = \dfrac{z}{p}, \dfrac{x^2}{m_1} = \dfrac{y^2}{n_1} = \dfrac{z^2}{p_1}.$

The second condition is equivalent to the existence of a real number l such that

$$\frac{m^2}{m_1} = \frac{n^2}{n_1} = \frac{p^2}{p_1} = 8l.$$

Replacing this relation into the first condition, we obtain

$$\frac{a}{4l} = m_2^2 - m_2, \frac{b}{4l} = n_2^2 - n_2, \frac{c}{4l} = p_2^2 - p_2.$$

where $m_2 = \dfrac{1}{m}, n_2 = \dfrac{1}{n}, p_2 = \dfrac{1}{p}$. So we infer that

$$1 = \frac{1}{m_2} + \frac{1}{n_2} + \frac{1}{p_2} = \frac{2\sqrt{l}}{\sqrt{l} + \sqrt{l+a}} + \frac{2\sqrt{l}}{\sqrt{l} + \sqrt{l+b}} + \frac{2\sqrt{l}}{\sqrt{l} + \sqrt{l+c}}$$

$$\Rightarrow \frac{1}{2\sqrt{l}} = \frac{1}{\sqrt{l} + \sqrt{l+a}} + \frac{1}{\sqrt{l} + \sqrt{l+b}} + \frac{1}{\sqrt{l} + \sqrt{l+c}}.$$

According to the definition of k, we must have $l = k$. Therefore, we conclude

$$x + y + z \geq m^{-m} n^{-n} p^{-p} m_1^{-am_1} n_1^{-bn_1} p_1^{-cp_1} = \frac{8l}{mnp} = 8l m_2 n_2 p_2$$

$$= \frac{1}{\sqrt{k}} \left(\sqrt{k} + \sqrt{k+a}\right) \left(\sqrt{k} + \sqrt{k+b}\right) \left(\sqrt{k} + \sqrt{k+c}\right).$$

with equality for

$$\frac{x}{m} = \frac{y}{n} = \frac{z}{p} = \frac{mnp}{am^2 + bn^2 + cp^2}.$$

2. This problem can be presented in another form as follows

★ *Let a, b, c, x, y, z be positive real numbers verifying $ax^2 + by^2 + cz^2 = xyz$.*

a. Prove that there exists exactly one positive number φ such that

$$\frac{2}{1 + \sqrt{1 + \varphi a}} + \frac{2}{1 + \sqrt{1 + \varphi b}} + \frac{2}{1 + \sqrt{1 + \varphi c}} = 1.$$

b. With this value of φ, prove that

$$x + y + z \geq \frac{(1 + \sqrt{1 + a\varphi})(1 + \sqrt{1 + \varphi b})(1 + \sqrt{1 + \varphi c})}{\varphi}.$$

3. Although it can't be denied that this general problem is helpful in creating particular inequalities (for particular values of a, b, c), we agree that the initial problem, created accidentally, not based on the general problem, is the most impressive (it has interesting coefficients $2, 3, 4, 5, 3, 2, 1$ and the expression attains its minimum when all variables a, b, c are equal to 6).

∇

Problem 82. *Let a, b, c be non-negative real numbers. Prove that*

$$\frac{1}{\sqrt{a^2+bc}}+\frac{1}{\sqrt{b^2+ca}}+\frac{1}{\sqrt{c^2+ab}} \geq \frac{2\sqrt{2}}{\sqrt{ab+bc+ca}}.$$

(Pham Kim Hung)

SOLUTION. First we may assume that $a \geq b \geq c$. Notice that

$$\frac{1}{\sqrt{b^2+ca}}+\frac{1}{\sqrt{c^2+ab}} \geq \frac{2\sqrt{2}}{\sqrt{b^2+c^2+ab+ac}},$$

so it suffices to prove that

$$\frac{1}{\sqrt{a^2+bc}}+\frac{2\sqrt{2}}{\sqrt{b^2+c^2+ab+ac}} \geq \frac{2\sqrt{2}}{\sqrt{ab+bc+ca}}.$$

Let $M = ab+bc+ca$ and $N = b^2+c^2+ab+ac$, then

$$\frac{2\sqrt{2}}{\sqrt{M}}-\frac{2\sqrt{2}}{\sqrt{N}} = \frac{2\sqrt{2}(b^2-bc+c^2)}{\sqrt{MN}\left(\sqrt{M}+\sqrt{N}\right)}.$$

Clearly, $N \geq M$; $N \geq 2(b^2-bc+c^2)$ and $M = ab+bc+ca \geq b\sqrt{a^2+bc}$, so

$$\frac{2\sqrt{2}(b^2-bc+c^2)}{\sqrt{MN}\left(\sqrt{M}+\sqrt{N}\right)} \leq \frac{2\sqrt{2}(b^2-bc+c^2)}{\sqrt{MN}\cdot 2\sqrt{M}} = \frac{\sqrt{2}(b^2-bc+c^2)}{M\sqrt{N}}$$

$$\leq \frac{\sqrt{2}(b^2-bc+c^2)}{b\sqrt{a^2+bc}\cdot\sqrt{2(b^2-bc+c^2)}} = \frac{\sqrt{b^2-bc+c^2}}{b\sqrt{a^2+bc}} \leq \frac{1}{\sqrt{a^2+bc}}.$$

So we are dwone. There is no case of equality.

∇

Problem 83. *Let a, b, c, d be positive real numbers with sum 4. Prove that*

$$\frac{1}{5-abc}+\frac{1}{5-bcd}+\frac{1}{5-cda}+\frac{1}{5-abc} \leq 1.$$

(Vasile Cirtoaje)

SOLUTION. Let $x = abc, y = bcd, z = cda, t = dab$. We need to prove that

$$\sum_{cyc}\frac{1}{5-x} \leq 1 \Leftrightarrow \sum_{cyc}\frac{1-x}{5-x} \geq 0 \Leftrightarrow \sum_{cyc}\frac{(1-x)(x+2)}{(5-x)(x+2)} \geq 0.$$

By **AM-GM** inequality it's easy to see that $x+y = bc(a+d) \leq \frac{64}{27} < 3$. So if $x \geq y$ then $(1-x)(2+x) \leq (1-y)(2+y)$ and $(5-x)(2+x) \geq (5-y)(2+y)$. According to **Chebyshev** inequality, we obtain

$$4\sum_{cyc}\frac{(1-x)(x+2)}{(5-x)(x+2)} \geq \left(\sum_{cyc}(1-x)(2+x)\right)\left(\sum_{cyc}\frac{1}{(5-x)(2+x)}\right).$$

It remains to prove that

$$\sum_{cyc}(1-x)(2+x) = 8 - \sum_{cyc} abc - \sum_{cyc} a^2b^2c^2 \geq 0.$$

First Solution. We let $p = a+b, q = ab, r = c+d$ and $s = cd$, then $p+r = 4$ and we need to prove that

$$A = sp + qr + s^2(p^2 - 2q) + q^2(r^2 - 2s) \leq 8.$$

Denote

$$A = f(q) = q^2(r^2 - 2s) + q(r - 2s^2) + sp + s^2p^2.$$

Since $f(q)$ is a convex function of q, we deduce that

$$f(q) \leq \max\left\{f(0), f\left(\frac{p^2}{4}\right)\right\}.$$

Similarly, if we consider A as a function of s, or $A = g(s)$, we obtain

$$g(s) \leq \max\left\{g(0), g\left(\frac{r^2}{4}\right)\right\}.$$

These two results combined show that A is maximum if and only if one of the numbers a, b, c, d equals 0 (case (1)) or $a = b, c = d$ (case (2)). Case (1) can be proved easily. In case (2), the inequality becomes

$$a^2c + c^2a + a^4c^2 + c^4a^2 \leq 4$$

Let $\beta = ac$, then $\beta \leq 1$ and

$$a^2c+c^2a+a^4c^2+c^4a^2 = 2ac+a^2c^2(4-2ac) = -2\beta^3+4\beta^2+2\beta = 4+(4-2\beta)(\beta^2-1) \leq 4.$$

This ends the proof. Equality holds for $a = b = c = d = 1$.

Second Solution. WLOG, suppose that $a \geq b \geq c \geq d$. Let $m = \dfrac{a+c}{2}, u = \dfrac{a-c}{2}$ and $t = m^2, v = u^2$ then we get

$$f(a,b,c,d) = \sum_{cyc} abc + \sum_{cyc} a^2b^2c^2 = g(v)$$

where

$$g(v) = (t-v)(b+d) + 2bd\sqrt{t} + (t-v)^2(b^2+d^2) + 2b^2d^2(t+v).$$

Since $t - v = ac \geq bd$, we deduce that

$$g'(v) = -(b+d) - 2(t-v)(b^2+d^2) + 2b^2d^2 \leq 0,$$

which implies $f(a,b,c,d) \le f(\sqrt{ac}, b, \sqrt{ac}, d)$. Now we repeat the procedure for the first two variables $(\sqrt{ac}, b)$ and then again for the first and the third. Repeating these procedures, and taking the limit, we conclude

$$f(a,b,c,d) \le f(\alpha,\alpha,\alpha,4-3\alpha),$$

for a certain $\alpha \in \left[0, \frac{4}{3}\right]$. The inequality $f(\alpha,\alpha,\alpha,4-3\alpha) \le 8$ is equivalent to

$$\alpha^3 + 3\alpha^2(4-3\alpha) + \alpha^6 + 3\alpha^4(4-3\alpha)^2 \le 8$$

$$\Leftrightarrow (\alpha-1)^2(7\alpha^4 - 4\alpha^3 - 3\alpha^2 - 4\alpha - 2) \le 0,$$

which is obvious because $\alpha \in \left[0, \frac{4}{3}\right]$ (and therefore $7\alpha^4 - 4\alpha^3 - 3\alpha^2 - 4\alpha - 2 \le 0$). The proof is finished and the equality occurs if $a = b = c = d = 1$.

Comment. By induction, we can prove the following result

★ *Suppose that n is a natural number greater than 3 and $a_1, a_2, ..., a_n$ are non-negative real numbers with sum n. For each number $k \in \{1, 2, ..., n\}$ we denote $b_k = a_1a_2...a_{k-1}a_{k+1}...a_n$. Prove that*

$$\frac{1}{n+1-b_1} + \frac{1}{n+1-b_2} + ... + \frac{1}{n+1-b_n} \le 1.$$

To prove it, we use induction for the following general inequality

$$\frac{1}{k-b_1} + \frac{1}{k-b_2} + ... + \frac{1}{k-b_n} \le \frac{n}{k-1},$$

where k is a real number and $k \ge n+1$. Notice that the most difficult step in solving this general problem is the proof of the case $n = 4$. For $n = 4$, the proof of $\sum_{cyc} \frac{1}{k-abc} \le \frac{4}{k-1}$ $(k \ge 5)$ can be obtained similarly to the case $k = 5$.

∇

Problem 84. *Let a, b, c be three arbitrary real numbers. Prove that*

$$\frac{1}{(2a-b)^2} + \frac{1}{(2b-c)^2} + \frac{1}{(2c-a)^2} \ge \frac{11}{7(a^2+b^2+c^2)}.$$

(Pham Kim Hung)

SOLUTION. Denote $x = 2a-b, y = 2b-c, z = 2c-a$. We get

$$a = \frac{4x+2y+z}{7}, b = \frac{4y+2z+x}{7}, c = \frac{4z+2x+y}{7},$$

$$\Rightarrow a^2+b^2+c^2=\frac{2(x+y+z)^2+x^2+y^2+z^2}{7}.$$

It remains to prove that for all $x,y,z\in\mathbb{R}$

$$\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\geq\frac{11}{2(x+y+z)^2+x^2+y^2+z^2}.$$

Certainly, we only need to consider the case $x\geq y\geq 0\geq z$ (and dismiss the case $x,y,z\geq 0$ or $x,y,z\leq 0$). In fact, we need to prove

$$f(x,y,z)=\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}-\frac{11}{2(x+y-z)^2+x^2+y^2+z^2}\geq 0$$

for all $x,y,z>0$ (that means we have changed the sign of z). Consider two cases

(i). The first case. If $z\geq x+y$, then it's easy to check that

$$\frac{1}{z-x-y}\left(f(x,y,z)-f(x,y,x+y)\right)=\frac{11(3z-x-y)}{M\cdot N}-\frac{x+y+z}{z^2(x+y)^2}$$

where

$$M=x^2+y^2+(x+y)^2\ \ ;\ \ N=x^2+y^2+z^2+2(z-x-y)^2\ \ ;$$

Since $3z-x-y\geq x+y+z, 2(x+y)^2\geq M$ and $5z^2\geq N$, it follows that

$$f(x,y,z)\geq f(x,y,x+y)=\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{(x+y)^2}-\frac{11}{2(x^2+y^2+(x+y)^2)}.$$

Notice that

$$\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{(x+y)^2}\right)\left(x^2+y^2+xy\right)-\frac{27}{4}$$
$$=\frac{x^2}{y^2}+\frac{y^2}{x^2}+\frac{x}{y}+\frac{y}{x}+\frac{x^2+xy+y^2}{(x+y)^2}-\frac{19}{4}\geq 0.$$

Therefore $f(x,y,x+y)\geq 0$ and we are done.

(ii). The second case. If $x+y\geq z$, setting $t=\sqrt{xy}$ we have

$$f(x,y,z)\geq f(t,t,z)=\frac{2}{t^2}+\frac{1}{z^2}-\frac{11}{2t^2+z^2+(2t-z)^2}.$$

WLOG, assume that $z=1$, then $t\leq\frac{1}{2}$. After expanding, the problem becomes

$$f(t)=5t^4-4t^3+6t^2-8t+3\geq 0.$$

Since $t\leq\frac{1}{2}$, $f'(t)=20t^3-12t^2+12t-8<(20t^3-10t^2)+(12t-6)\leq 0$ and therefore we can conclude that

$$f(t)\geq f\left(\frac{1}{2}\right)=0.3125>0.$$

Comment. The best constant k for which the following inequality

$$\frac{1}{(2a-b)^2}+\frac{1}{(2b-c)^2}+\frac{1}{(2c-a)^2}\ge\frac{k}{7(a^2+b^2+c^2)}$$

is true for all $a,b,c\in\mathbb{R}$ is $k=\min\limits_{0\le x\le 1/2} g(x)=10x^2+\frac{6}{x^2}-\frac{16}{x}-8x+23.$

Afrer some calculations, we find this value to be approximately 11.6075.

∇

Problem 85. *Let a,b,c be positive real numbers. Consider the following inequality*

$$\frac{ab}{c}+\frac{bc}{a}+\frac{ca}{b}\ge 3\sqrt[k]{\frac{a^k+b^k+c^k}{3}}\quad(\star)$$

(a). Prove that $(\star)$ is true for $k=2$.

(b). Prove that $(\star)$ is not true for $k=3$ but true for $k=3$ and a,b,c such that

$$a^3b^3+b^3c^3+c^3a^3\ge abc(a^3+b^3+c^3).$$

(c). For which value of k is $(\star)$ true for all positive real numbers a,b,c?

•

(Pham Kim Hung)

SOLUTION. (a). For $k=2$, we can assume that $\sum\limits_{cyc}a^2=3$ without loss of generality. The inequality $\sum\limits_{cyc}\frac{ab}{c}\ge 3$ is equivalent to

$$\left(\sum_{cyc}\frac{ab}{c}\right)^2\ge 9\Leftrightarrow\sum_{cyc}\frac{a^2b^2}{c^2}\ge 3\Leftrightarrow\sum_{cyc}a^2\left(\frac{b^2}{c^2}+\frac{c^2}{b^2}-2\right)\ge 0,$$

which is obvious. Equality holds for $a=b=c$.

(b). If $k=3$, we let $a=b=0.8$ and $c=\sqrt[3]{1.976}$. Then $(\star)$ is not true. Now, with the condition $a^3b^3+b^3c^3+c^3a^3\ge abc(a^3+b^3+c^3)$, $(\star)$ becomes true. Indeed, according to AM-GM inequality, with the supposition that $\sum\limits_{cyc}a^3=3$, we obtain

$$\left(\sum_{cyc}a\right)\left(\sum_{cyc}a^2\right)=3+\sum_{cyc}a^2(b+c)=\sum_{cyc}(a^2b+b^2a+1)\ge 3\sum_{cyc}ab$$

$$\Rightarrow abc\left(\sum_{cyc}a^3+\sum_{cyc}ab(a+b)\right)\ge 3abc\left(\sum_{cyc}ab\right).$$

Because $abc\sum_{cyc} a^3 \leq \sum_{cyc} a^3b^3$, we deduce that

$$abc\left(\sum_{cyc} ab(a+b)\right) + \sum_{cyc} a^3b^3 \geq 3abc\left(\sum_{cyc} ab\right)$$

$$\Rightarrow \left(\sum_{cyc} ab\right)\left(\sum_{cyc} a^2b^2\right) \geq 3abc\left(\sum_{cyc} ab\right) \Rightarrow \sum_{cyc} a^2b^2 \geq 3abc \Rightarrow \sum_{cyc} \frac{ab}{c} \geq 3.$$

We are done. Equality holds for $a = b = c$.

(c). Consider that a, b, c are positive real numbers verifying (normalization)

$$\frac{ab}{c} + \frac{bc}{a} + \frac{ca}{b} = 3.$$

Let's find the maximum of the expression

$$S = a^k + b^k + c^k, \text{ for } k > 0.$$

Let $x = \dfrac{ab}{c}, y = \dfrac{bc}{a}, z = \dfrac{ca}{b}$, then $x + y + z = 3$ and $S = \sum_{cyc}(xy)^{k/2}$. From a known result (see one of the previous problems)

$$S \leq \max\left\{3, \frac{3^k}{2^k}\right\}.$$

It's easy to conclude that $(\star)$ is true for all positive real numbers a, b, c if and only if $k \leq \dfrac{\ln 3}{\ln 3 - \ln 2} \approx 2.709511... < 3$ (including case $k < 0$ of course).

∇

Problem 86. *Let a, b, c be non-negative real numbers. Prove that*

$$\frac{1}{\sqrt{4a^2+bc}} + \frac{1}{\sqrt{4b^2+ca}} + \frac{1}{\sqrt{4c^2+ab}} \geq \frac{4}{a+b+c}.$$

(Pham Kim Hung)

SOLUTION. **First Solution.** We denote

$$S = \sum_{cyc} \frac{1}{\sqrt{4a^2+bc}} \quad ; \quad P = \sum_{cyc}(b+c)^3(4a^2+bc).$$

According to **Hölder** inequality, we deduce that

$$S \cdot S \cdot P \geq (a+b+c)^3.$$

So it's enough to prove that $(a+b+c)^5 \geq 2P$. Since

$$P = \sum_{cyc} a^4(b+c) + 7\sum_{cyc} a^3(b^2+c^2) + 24abc\sum_{cyc} ab,$$

$$(a+b+c)^5 = \sum_{cyc} a^5 + 5\sum_{cyc} a^4(b+c) + 10\sum_{cyc} a^3(b^2+c^2) + 20abc\sum_{cyc} a^2 + 30abc\sum_{cyc} ab,$$

the inequality $(a+b+c)^5 \geq 2P$ is equivalent to (after reducing similar terms)

$$\sum_{cyc} a^5 + 3\sum_{cyc} a^4(b+c) + 20abc\sum_{cyc} a^2 \geq 4\sum_{cyc} a^3(b^2+c^2) + 18abc\sum_{cyc} ab.$$

This last inequality can be deduced from the following results

$$18abc\sum_{cyc} a^2 \geq 18abc\sum_{cyc} ab,$$

$$\sum_{cyc} a^5 + abc\sum_{cyc} a^2 \geq \sum_{cyc} a^4(b+c),$$

$$4\sum_{cyc} a^4(b+c) \geq 4\sum_{cyc} a^3(b^2+c^2).$$

We are done. The equality holds for $a=b, c=0$ or permutations.

Second Solution. Suppose that $a \geq b \geq c$. Denote $t = \dfrac{a+b}{2} \geq c$, then the inequality

$$(4a^2+bc)(4b^2+ca) \leq (4t^2+tc)^2$$

is equivalent to

$$(a-b)^2\left(\frac{1}{4}c^2 + a^2 + b^2 + 6ab - 3ca - 3cb\right) \geq 0,$$

which is clearly true because $a \geq b \geq c$. We deduce that

$$\frac{1}{\sqrt{4a^2+bc}} + \frac{1}{\sqrt{4b^2+ca}} + \frac{1}{\sqrt{4c^2+ab}} \geq \frac{2}{\sqrt{4t^2+tc}} + \frac{1}{\sqrt{4c^2+t^2}}.$$

It remains to prove that

$$\frac{2}{\sqrt{4t^2+tc}} + \frac{1}{\sqrt{4c^2+t^2}} \geq \frac{4}{a+b+c} = \frac{4}{2t+c}$$

$$\Leftrightarrow \left(\frac{2}{\sqrt{4t^2+tc}} - \frac{1}{t}\right) + \left(\frac{1}{\sqrt{4c^2+t^2}} - \frac{1}{t}\right) \geq \frac{4}{2t+c} - \frac{2}{t}$$

$$\Leftrightarrow \frac{-c}{\sqrt{4t^2+tc}(2t+\sqrt{2t^2+tc})} + \frac{-4c^2}{t\sqrt{4c^2+t^2}(t+\sqrt{4c^2+t^2})} \geq \frac{-2c}{t(2t+c)}$$

$$\Leftrightarrow \frac{2}{t(2t+c)} - \frac{1}{\sqrt{4t^2+tc}(2t+\sqrt{2t^2+tc})} - \frac{4c}{t\sqrt{4c^2+t^2}(t+\sqrt{4c^2+t^2})} \geq 0.$$

Notice that

$$\frac{1}{3t(2t+c)} \geq \frac{1}{\sqrt{4t^2+tc}(2t+\sqrt{2t^2+tc})} \quad (\star)$$

$$\Leftrightarrow \; 9t^2(2t+c)^2 \leq (4t^2+tc)(2t+\sqrt{4t^2+tc})^2$$

$$\Leftrightarrow \; t^2+6tc+2c^2 \leq 2t\sqrt{4t^2+tc}+c\sqrt{4t^2+tc},$$

which is obvious because $t \geq c$. We will now prove that

$$\frac{5}{3t(2t+c)} \geq \frac{4c}{t\sqrt{4c^2+t^2}(t+\sqrt{4c^2+t^2})} \quad (\star\star)$$

$$\Leftrightarrow \; 5\sqrt{4c^2+t^2}(t+\sqrt{4t^2+c^2}) \geq 12c(2t+c).$$

According to **Cauchy-Schwarz** inequality, we deduce that $\sqrt{5(4c^2+t^2)} \geq 4c+t$ and similarly, $\sqrt{5(4t^2+c^2)} \geq 4t+c$. So it's enough to prove that

$$(4c+t)(4c+(\sqrt{5}+1)t) \geq 12c(2t+c)$$

$$\Leftrightarrow \; (\sqrt{5}+1)t^2+(16-4\sqrt{5})tc+4c^2 \geq 0.$$

which is obvious. Combining the results $(\star)$ and $(\star\star)$ we get the desired result.

∇

Problem 87. *Let a, b, c be non-negative real numbers. Prove that*

$$\sqrt{\frac{ab}{4a^2+b^2+4c^2}}+\sqrt{\frac{bc}{4b^2+c^2+4a^2}}+\sqrt{\frac{ca}{4c^2+a^2+4b^2}} \leq 1.$$

(Pham Kim Hung)

SOLUTION. WLOG, suppose that $a^2+b^2+c^2=3$. By the weighted **Jensen** inequality, we deduce that

$$\sum_{cyc}\sqrt{\frac{ab}{4a^2+b^2+4c^2}} = \sum_{cyc}\frac{a^2+4b^2+4c^2}{27}\cdot\sqrt{\frac{27^2\cdot ab}{(4a^2+b^2+4c^2)(a^2+4b^2+4c^2)^2}}$$

$$\leq \sqrt{\sum_{cyc}\frac{27ab}{(4a^2+b^2+4c^2)(a^2+4b^2+4c^2)}} = \sqrt{\sum_{cyc}\frac{3ab}{(4-a^2)(4-b^2)}}.$$

It remains to prove that

$$3\sum_{cyc}ab(4-c^2) \leq \prod_{cyc}(4-a^2)$$

$$\Leftrightarrow \; 4\left(\sum_{cyc}ab\right)\left(\sum_{cyc}a^2\right) \leq \frac{16}{9}\left(\sum_{cyc}a^2\right)^2+4\sum_{cyc}a^2b^2+3\sum_{cyc}a^2bc-a^2b^2c^2$$

$$\Leftrightarrow\ 36\sum_{cyc} a^3(b+c) + 9\sum_{cyc} a^2bc + 9a^2b^2c^2 \le 16\sum_{cyc} a^4 + 68\sum_{cyc} a^2b^2.$$

Because $3abc\sum_{cyc} a = abc\left(\sum_{cyc} a\right)\left(\sum_{cyc} a^2\right) \ge 9a^2b^2c^2$, we only need to prove that

$$9\sum_{cyc} a^3(b+c) + 3\sum_{cyc} a^2bc \le 4\sum_{cyc} a^4 + 17\sum_{cyc} a^2b^2$$

$$\Leftrightarrow\ \sum_{cyc}\left(2a^2 + 2b^2 + \frac{3c^2}{2} - 5ab\right)(a-b)^2 \ge 0.$$

Suppose that $a \ge b \ge c$, then we are done by Abel's inequality because $2b^2 + 2c^2 + \frac{3a^2}{2} - 5bc \ge 0$ and

$$\left(2a^2 + 2b^2 + \frac{3c^2}{2} - 5ab\right) + \left(2a^2 + 2c^2 + \frac{3b^2}{2} - 5ac\right) = 4a^2 + \frac{9}{2}(b^2+c^2) - 5a(b+c)$$

$$\ge 4a^2 + 9\left(\frac{b+c}{2}\right)^2 - 10a\left(\frac{b+c}{2}\right) \ge 0.$$

∇

Problem 88. *Suppose that n is a positive integer and $(x_1, x_2, .., x_n)$; $(y_1, y_2, ..., y_n)$ are two positive real sequences. Let $(z_2, z_3, ..., z_{2n})$ be a positive sequence satisfying*

$$z_{i+j}^2 \ge x_i y_j\ \forall 1 \le i, j \le n.$$

Denote $M = \max\{z_2, ..., z_{2n}\}$. Prove the following inequality

$$\left(\frac{M + z_2 + ... + z_{2n}}{2n}\right)^2 \ge \left(\frac{x_1 + x_2 + ... + x_n}{n}\right) \cdot \left(\frac{y_1 + y_2 + ... + y_n}{n}\right).$$

(IMO Shortlist 2003)

Solution. Let $X = \max\{x_1, x_2, ..., x_n\}$ and $Y = \max\{y_1, y_2, ..., y_n\}$. WLOG, we can assume that $X = Y = 1$ (otherwise, replace x_i with $x_i/X, y_i$ with y_i/Y and z_i with $z_i/\sqrt{XY}$). According to AM-GM inequality, the following result is stronger

$$M + z_2 + ... + z_{2n} \ge x_1 + x_2 + ... + x_n + y_1 + y_2 + ... + y_n\ (\star)$$

Let r be a certain real numbers. We will prove that the number of terms of the right-hand expression of $(\star)$ which are bigger than r is not bigger than the number of terms on the left-hand expression of $(\star)$ which are bigger than r. In fact, this clause is clearly

true if $r > 1$ (because there are no terms on the right-hand expression of $(\star)$ bigger than r). Consider now the case $r < 1$. We denote

$$A = \{i \in N,\ 1 \le i \le n \mid x_i > r\},$$

$$B = \{i \in N,\ 1 \le i \le n \mid y_i > r\},$$

then certainly $|A|, |B| \ge 1$. Suppose that $A = \{i_1, i_2, ..., i_a\}$ and $B = \{j_1, j_2, ..., j_b\}$ with $i_1 < i_2 < ... < i_a$ and $j_1 < j_2 < ... < j_b$. There are at least $a + b - 1$ terms of the sequence $(z_2, z_3, ..., z_{2n})$ which are bigger than r as:

$$z_{i_1+j_1}, z_{i_1+j_2}, ..., z_{i_1+j_b}, z_{i_2+j_b}, ..., z_{i_a+j_b}.$$

On the other hand, notice that $a + b - 1 \ge 1$, so at least one number z_i is bigger than r, so $M > r$. This implies that there are at least $a + b$ terms on the left-hand expression of $(\star)$ bigger than r.

From this property, we conclude that for every natural number k $(1 \le k \le 2n)$, the k^{th}–greatest number of the left-hand expression of $(\star)$ is not smaller than the k^{th}–greatest number of the right-hand expression of $(\star)$. So, obviously, the sum of all terms of the left hand expression of $(\star)$ is not smaller than that of the right-hand expression of $(\star)$. The problem is completely solved.

∇

Problem 89. *(a).Let a, b, c be three real numbers. Prove that*

$$a^4 + b^4 + c^4 + ab^3 + bc^3 + ca^3 \ge 2(a^3b + b^3c + c^3a).$$

(b). Let a, b, c be three real numbers and $a^2 + b^2 + c^2 + ab + bc + ca = 6$. Prove that

$$a^3b + b^3c + c^3a + abc(a + b + c) \le 6.$$

(c). Find the best (greatest) constant k such the the following inequality holds for all real numbers a, b, c

$$a^4 + b^4 + c^4 + k(ab + bc + ca)^2 \ge (1 + 3k)(a^3b + b^3c + c^3a)$$

(Vasile Cirtoaje and Pham Kim Hung)

SOLUTION. For all real numbers a, b, c, we have that

$$(a^2 - kab + kac - c^2)^2 + (b^2 - kbc + kba - a^2)^2 + (c^2 - kca + kcb - b^2)^2 \ge 0.$$

After expanding, this inequality becomes

$$\sum_{cyc} a^4 + (k^2-1)\sum_{cyc} a^2b^2 + k\sum_{cyc} ab^3 \ge 2k\sum_{cyc} a^3b + (k^2-k)\sum_{cyc} a^2bc.$$

(a). Let $k=1$. We obtain

$$a^4+b^4+c^4+ab^3+bc^3+ca^3 \ge 2(a^3b+b^3c+c^3a).$$

(b). Let $k=2$, we obtain

$$\left(\sum_{cyc} a^2\right)^2 + \sum_{cyc} a^2b^2 + \sum_{cyc} ab^3 \ge 4\sum_{cyc} a^3b + 2abc\sum_{cyc} a$$

$$\Leftrightarrow \left(\sum_{cyc} a^2 + \sum_{cyc} ab\right)^2 \ge 6\sum_{cyc} a^3b + 6abc\sum_{cyc} a.$$

If $a^2+b^2+c^2+ab+bc+ca=6$ then $a^3b+b^3c+c^3a+abc(a+b+c) \le 6$.

(c). We have

$$\left((a-b)^2+2c(a-c)\right)^2 + \left((b-c)^2+2a(b-a)\right)^2 + \left((c-a)^2+2b(c-b)\right)^2 \ge 0.$$

After expanding, this inequality becomes

$$6\sum_{cyc} a^4 + 4\sum_{cyc} a^2bc + 2\sum_{cyc} a^2b^2 \ge 12\sum_{cyc} a^3b$$

$$\Rightarrow \sum_{cyc} a^4+b^4+c^4+\frac{1}{3}\left(\sum_{cyc} ab\right)^2 \ge 2\sum_{cyc} a^3b.$$

So the best constant k (greatest) that we are looking for is $k=1/3$.

Comment. In all these results, there is a special case of equality different from the trivial case $a=b=c$. For example, in part (a), the equality holds for

$$a = 2\cos 20 + 1 \approx 2.88, b = 2\cos 40 \approx 1.532, c = -1.$$

∇

Problem 90. *Let a,b,c,d be non-negative real numbers verifying $a+b+c+d=4$. Prove that for all positive integers k,n greater than 2*

$$(k+a^n)(k+b^n)(k+c^n)(k+d^n) \ge (k+1)^4.$$

(Pham Kim Hung)

SOLUTION. Notice that for all $n \geq 2$, we have

$$\left(\frac{k+a^n}{k+1}\right)^2 \geq \left(\frac{k+a^2}{k+1}\right)^n.$$

This inequality can be proved easily by **AM-GM** inequality or **Hölder** inequality. According to this result, we get

$$\prod_{cyc}\left(\frac{k+a^n}{k+1}\right)^2 \geq \prod_{cyc}\left(\frac{k+a^2}{k+1}\right)^n.$$

Therefore it's enough to prove the inequality for $n = 2$. Suppose that this inequality is true for $k = 2$, then it will be true for all $k \geq 2$ because **Hölder** inequality claims that

$$\prod_{cyc}(k+a^2) = \prod_{cyc}\left((k-2)+(2+a^2)\right) \geq \left((k-2)+\sqrt[4]{\prod_{cyc}(2+a^2)}\right)^4 \geq (k+1)^4.$$

So it suffices to prove the inequality in the case $k = n = 2$, namely

$$(2+a^2)(2+b^2)(2+c^2)(2+d^2) \geq 81.$$

First Solution. *(symmetric separation)* The inequality is equivalent to

$$\sum_{cyc}\ln(2+a^2) \geq 4\ln 3.$$

Consider the following function

$$f(x) = \ln(2+x^2) - \ln 3 - \frac{2x}{3} + \frac{2}{3}.$$

Its derivative is

$$f'(x) = \frac{2x}{2+x^2} - \frac{2}{3} = \frac{2}{3}(x-1)(2-x).$$

So $f(x)$ is decreasing on $[0,1]$ and $[2,+\infty]$, increasing on $[1,2]$, therefore

$$\min_{0\leq x\leq t} f(x) = \min\{f(1), f(t)\}\ \forall t \in [0,4]\ (\star)$$

From $(\star)$, we deduce that for all $1 \leq x \leq 2.5$, $f(x) \geq f(1) = 0$. If all numbers a, b, c, d are smaller than 2.5 then we are done because

$$\sum_{cyc} f(a) \leq 0 \Leftrightarrow \sum_{cyc}\ln(2+x^2) \leq \left(\frac{2x}{3} - \frac{2}{3} + \ln 3\right) = 4\ln 3.$$

Otherwise, suppose that $a \geq 2.5$. Let $t = \frac{1}{3}(b+c+d)$ then clearly

$$\prod_{cyc}(2+a^2) \geq 16 + 8\sum_{cyc} a^2 + 4a^2(b^2+c^2+d^2) \geq 16 + 8a^2 + 24t^2 + 12a^2t^2.$$

It remains to prove that for all $t \le 0.5$

$$g(t) = 8(4-3t)^2 + 24t^2 + 12t^2(4-3a)^2 \ge 65.$$

Since $4 - 3t \ge 2.5$ and $t(4-3t)^2 < 4$

$$g'(t) = -48(4-3t)+48t+24t(4-3t)^2-72t^2(4-3t) \le -48\cdot(2.5)^2+48\cdot(0.5)+24\cdot 4 < 0,$$

and we can conclude that

$$g(t) \ge g(0.5) = 74.75 > 65.$$

The proof is completed and equality holds for $a = b = c = d = 1$.

Second Solution. Notice that if $a + b \le 2$ then

$$(2+a^2)(2+b^2) \ge \left(2 + \frac{(a+b)^2}{4}\right)^2.$$

Indeed, this is equivalent to

$$2(a^2+b^2) - (a+b)^2 - \frac{(a+b)^4}{16} - a^2b^2 \ge 0 \Leftrightarrow (a-b)^2\left(16 - (a+b)^2 - 4ab\right) \ge 0$$

which is obvious because $a + b \le 2$. Now suppose that $d \ge c \ge b \ge a$ and

$$F(a,b,c,d) = \prod_{cyc}(2+a^2).$$

Because $c + a \le 2$, so $F(a,b,c,d) \ge F\left(\frac{a+c}{2}, b, \frac{a+c}{2}, d\right)$. As in the second solution of problem 83,

$$F(a,b,c,d) \ge F(x,x,x,4-3x),\ x = \frac{a+b+c}{3}.$$

It remains to prove that $(2+x^2)^3\left(2+(4-3x)^2\right) \ge 81$, or $f(x) \ge 4\ln 3$ where

$$f(x) = 3\ln(2+x^2) + \ln\left(2+(4-3x)^2\right).$$

It's easy to check that

$$f'(x) = \frac{6x}{2+x^2} - \frac{6(4-3x)}{2+(4-3x)^2}$$

and $f'(x) = 0$ if and only if

$$(x-1)\left(4 - \frac{8}{x(4-3x)}\right) = 0.$$

If $x \neq 1$, we must have $x(4-3x) = 2$. However, **AM-GM** inequality shows that $3x(4-3x) \le 4$, or $x(4-3x) \le 4/3 < 2$. So the equation $f'(x) = 0$ has exactly one positive real root $x = 1$. Therefore we conclude that

$$\max_{0\le x\le 1} f(x) = f(1) = 4\ln 3.$$

Comment. By a similar method as in the second solution, we can propose and solve a general problem as follows

★ *Let a, b, c, d be non-negative real numbers with sum 4. For all $k \ge 1$, prove that*

$$(k+a^2)(k+b^2)(k+c^2)(k+d^2) \ge \min\left\{(k+1)^4; (k+\alpha^2)^3\left(k+(4-3\alpha)^2\right)\right\},$$

where α is determined in the case $k \le \frac{4}{3}$ as $\alpha = \dfrac{2-\sqrt{4-3k}}{3}$.

To prove this, just notice that if $k \ge 1$ and $a + b \le 2$ then

$$(k+a^2)(k+b^2) \ge \left(k + \frac{(a+b)^2}{4}\right)^2.$$

By choosing particular cases k's, we can obtain interesting results as follows

$$(5+4a^2)(5+4b^2)(5+4c^2)(5+4d^2) \ge 6480.$$

$$(1+a^2)(1+b^2)(1+c^2)(1+d^2) \ge 10\left(1+\frac{1}{9}\right)^3 = \frac{10^4}{9^3}.$$

$$(4+3a^2)(4+3b^2)(4+3c^2)(4+3d^2) \ge \min\left(7^4, \frac{2^{16}}{3^3}\right) = 7^4 = 2401.$$

∇

Problem 91. *Let a, b, c be positive real numbers such that $a^2+b^2+c^2 = 3$. Prove that*

$$a^3b^2 + b^3c^2 + c^3a^2 \le 3.$$

Solution. By **Cauchy-Schwarz** inequality, we have

$$(a^3b^2+b^3c^2+c^3a^2)^2 \le (a^2b^2+b^2c^2+c^2a^2)(a^4b^2+b^4c^2+c^4a^2).$$

It remains to prove that if $x+y+z=3$ then

$$(xy+yz+zx)(x^2y+y^2z+z^2x) \le 3.$$

Notice that

$$3(xy+yz+zx)(x^2y+y^2z+z^2x)=\left(\sum_{cyc}x\right)\left(\sum_{cyc}xy\right)\left(\sum_{cyc}x^2y\right)$$

$$=\left(\sum_{cyc}xy\right)\left(\sum_{cyc}x^3y+\sum_{cyc}x^2y^2+3xyz\right).$$

Let $s=xy+yz+zx$ then $3abc\geq 4s-9$ by **Schur** inequality. Moreover

$$\sum_{cyc}x^2=9-2s\ ;\ \sum_{cyc}x^2y^2=s^2-6xyz\ ;$$

Taking into account problem 52, we have

$$3\sum_{cyc}x^3y\leq(x^2+y^2+z^2)^2=(9-2s)^2.$$

We deduce that

$$3\left(\sum_{cyc}xy\right)\left(\sum_{cyc}x^3y+\sum_{cyc}x^2y^2+3xyz\right)$$

$$\leq s\left((9-2s)^2+3s^2-9abc\right)\leq s\left((9-2s)^2+3s^2-3(4s-9)\right).$$

It suffices to prove that

$$s\left((9-2s)^2+3s^2-3(4s-9)\right)\leq 81\ \Leftrightarrow\ (s-3)(7s^2-27s+27)\leq 0,$$

which is obvious because $s\leq 3$. We are done and the equality holds for $a=b=c$.

Comment. According to this result, we can easily obtain that (due to the *Cauchy reverse*)

★ *Given positive real numbers with sum* 3. *Prove that*

$$\frac{a}{1+b^3}+\frac{b}{1+c^3}+\frac{c}{1+a^3}\geq\frac{3}{2}.$$

∇

Problem 92. *Let a,b,c be arbitrary positive real numbers. Prove that*

$$\left(a+\frac{b^2}{c}\right)^2+\left(b+\frac{c^2}{a}\right)^2+\left(c+\frac{a^2}{b}\right)^2\geq\frac{12(a^3+b^3+c^3)}{a+b+c}.$$

(Pham Kim Hung)

SOLUTION. The inequality is equivalent to

$$a^2+b^2+c^2+\frac{2ab^2}{c}+\frac{2bc^2}{a}+\frac{2ca^2}{b}+\frac{b^4}{c^2}+\frac{c^4}{a^2}+\frac{a^4}{b^2}\ge\frac{12(a^3+b^3+c^3)}{a+b+c}.$$

Using the following identities

$$\sum_{cyc}\frac{b^4}{c^2}-\sum_{cyc}a^2=\sum_{cyc}(b-c)^2\left(1+\frac{b}{c}\right)^2,$$
$$\sum_{cyc}\frac{ab^2}{c}-\sum_{cyc}ab=\sum_{cyc}\frac{a(b-c)^2}{c},$$
$$\frac{3\sum\limits_{cyc}a^3}{\sum\limits_{cyc}a}-\sum_{cyc}a^2=\frac{\sum\limits_{cyc}(b+c)(b-c)^2}{a+b+c},$$

the inequality can be rewritten as

$$\sum_{cyc}(b-c)^2\left(\left(1+\frac{b}{c}\right)^2-1-\frac{4(b+c)}{a+b+c}+\frac{2a}{c}\right)\ge 0$$

or $S_a(b-c)^2+S_b(c-a)^2+S_c(a-b)^2\ge 0$ where

$$S_a=\frac{b^2}{c^2}+\frac{4a}{a+b+c}+\frac{2(a+b)}{c}-4,$$
$$S_b=\frac{c^2}{a^2}+\frac{4b}{a+b+c}+\frac{2(b+c)}{a}-4,$$
$$S_c=\frac{a^2}{b^2}+\frac{4c}{a+b+c}+\frac{2(a+c)}{b}-4.$$

(i). The first case. $c\ge b\ge a$. Clearly, $S_b\ge 0$ and

$$\begin{aligned}S_a+S_b&=\frac{c^2}{a^2}+\frac{b^2}{c^2}+\frac{4(a+b)}{a+b+c}+\frac{2(b+c)}{a}+\frac{2(a+b)}{c}-8\\&>\left(\frac{c^2}{b^2}+\frac{b^2}{c^2}-4\right)+\left(\frac{2c}{a}+\frac{2a}{c}-4\right)+\left(\frac{2b}{a}-2\right)\ge 0.\end{aligned}$$

Similarly, we have that

$$\begin{aligned}S_c+S_b&=\frac{c^2}{a^2}+\frac{a^2}{b^2}+\frac{4(b+c)}{a+b+c}+\frac{2(b+c)}{a}+\frac{2(a+c)}{b}-8\\&>\left(\frac{c^2}{a^2}+\frac{a^2}{b^2}-2\right)+\left(\frac{2b}{a}+\frac{2a}{b}-4\right)+\left(\frac{2c}{a}-2\right)\ge 0.\end{aligned}$$

So the desired result follows because

$$S_a(b-c)^2+S_b(a-c)^2+S_c(a-b)^2\ge(S_a+S_b)(b-c)^2+(S_b+S_c)(a-b)^2\ge 0.$$

(ii). The second case. $a \geq b \geq c$. Clearly, $S_a \geq 1$, $S_c \geq -1 + \dfrac{4c}{a+b+c}$ and

$$\begin{aligned}S_a + 2S_b &= \frac{2c^2}{a^2} + \frac{b^2}{c^2} + \frac{8b+4a}{a+b+c} + \frac{4(b+c)}{a} + \frac{2(a+b)}{c} - 12\\ &> \left(\frac{8b+4a}{a+b+c} - 4\right) + \left(\frac{2b}{a} + \frac{2a}{c} - 4\right) + \left(\frac{2c}{a} + \frac{2a}{c} - 4\right) \geq 0.\end{aligned}$$

If $2b \geq a + c$ then we have

$$\begin{aligned}S_a + 4S_b + S_c &\geq \frac{4c^2}{a^2} + \frac{b^2}{c^2} + \frac{16b+4a+4c}{a+b+c} + \frac{8(b+c)}{a} + \frac{2(a+b)}{c} - 21\\ &\geq \frac{4c^2}{a^2} + \frac{b^2}{c^2} + \frac{8(b+c)}{a} + \frac{2(a+b)}{c} - 13\\ &\geq \frac{4c^2}{a^2} + \frac{16c}{a} + \frac{2a}{c} - 10 \geq 2\sqrt{32} - 10 \geq 0.\end{aligned}$$

Consider the cases

If $a + c \leq 2b$, certainly $2(b - c) \geq a - c$. If $S_b \geq 0$ we are done immediately. Otherwise, suppose that $S_b \leq 0$, then

$$S_a(b-c)^2 + S_b(a-c)^2 + S_c(a-b)^2 \geq (S_a + 4S_b + S_c)(b-c)^2 \geq 0.$$

If $a + c \geq 2b$, we will prove $S_c + 2S_b \geq 0$, or

$$g(c) = \frac{2c^2}{a^2} + \frac{a^2}{b^2} + \frac{8b+4c}{a+b+c} + \frac{4(b+c)}{a} + \frac{2(a+c)}{b} - 12 \geq 0.$$

Just notice that $g(c)$ is an increasing function of $c \geq 0$ and $c \geq 2b - a$, therefore

(.) If $a \geq 2b$, we have that

$$g(c) \geq g(0) = \frac{a^2}{b^2} + \frac{8b}{a+b} + \frac{4b}{a} + \frac{2a}{b} - 12$$

$$= \left(\frac{a+b}{b} + \frac{9b}{a+b} - 6\right) + \left(\frac{a}{b} + \frac{4b}{a} - 4\right) + \left(\frac{a^2}{b^2} - 4\right) + \left(\frac{1}{3} - \frac{b}{a+b}\right) + \frac{2}{3} \geq 0.$$

(.) If $a \leq 2b$, it's easy to infer that

$$g(c) \geq g(2b-a) = \frac{8b^2}{a^2} + \frac{a^2}{b^2} + \frac{4b}{a} - \frac{4a}{3b} - \frac{14}{3} \geq 0.$$

We obtain the conclusion because

$$S_a(b-c)^2 + S_b(a-c)^2 + S_c(a-b)^2 \geq (S_a + 2S_b)(b-c)^2 + (S_c + 2S_b)(a-b)^2 \geq 0.$$

∇

Problem 93. *Suppose that n is an integer greater than 2 and $a_1, a_2, ..., a_n$ are n real numbers. Prove that for any non-empty subset S of $\{1, 2, ..., n\}$, we have*

$$\left(\sum_{i\in S} a_i\right)^2 \le \sum_{1\le i\le j\le n} (a_i + ... + a_j)^2.$$

(Gabriel Dospinescu)

SOLUTION. We will first prove the following lemma

Lemma. For all real numbers $x_1, x_2, ..., x_{2k+1}$

$$\left(\sum_{0\le i\le k} a_{2i+1}\right)^2 \le \sum_{1\le i\le j\le 2k+1} (a_i + ... + a_j)^2 \ (\star)$$

PROOF. Let $s_i = x_1 + x_2 + ... + x_i \ \forall i \in \{1, 2, ..., k\}$, then

$$\sum_{0\le i\le k} a_{2i+1} = s_1 + s_3 - s_2 + s_5 - s_4 + ... + s_{2k+1} - s_{2k}.$$

The left-hand expression of $(\star)$ can be rewritten to

$$\sum_{i=1}^{2k+1} s_i^2 - 2\sum_{i,j} s_{2i}s_{2j+1} + 2\sum_{i<j} s_{2i}s_{2j} + 2\sum_{i<j} s_{2i+1}s_{2j+1} + 2,$$

and the right-hand expression of $(\star)$ is

$$\sum_{i<j}(s_i - s_j)^2 = (2k+1)\sum_{i=1}^{2k+1} s_i^2 - 2\sum_{i<j} s_i s_j.$$

After reducing similar terms on both sides, we have the following equivalent inequality

$$2k\sum_{i=1}^{2k+1} s_i^2 \ge 4\sum_{i<j} s_{2i}s_{2j} + 4\sum_{i<j} s_{2i+1}s_{2j+1},$$

$$\Leftrightarrow \sum_{1\le i<j\le k} (s_{2i} - s_{2j})^2 + \sum_{1\le i<j\le k} (s_{2i+1} - s_{2j+1})^2 + \sum_{1\le i<j\le k} s_{2i}^2 \ge 0,$$

which is obvious. The lemma is completely solved.

Returning to our original problem, we have to prove it for an arbitrary set $S \subset \{1, 2, ..., n\}$. Obviously, if S has no separated elements ($S = \{i, i+1, ..., j\}$) we are done (because all terms of the right-hand expression appear in the left-hand expression. Suppose that S is formed by some separated "segments", namely

$$S = \{j_1, j_1 + 1, ..., j_2, j_3, j_3 + 1 ..., j_4, ..., j_{2m+1}, j_{2m+1} + 1, ..., j_{2m+2}\}.$$

Denote

$$b_1 = a_{j_1} + a_{j_1+1} + \dots + a_{j_2}$$
$$b_2 = a_{j_2+1} + \dots + a_{j_3}$$
$$\dots\dots$$
$$b_{2k+1} = a_{j_{2m+1}} + a_{j_{2m+1}+1} + \dots + a_{j_{2m+2}}$$

According to the previous lemma, with the remark that $\sum_{i\in S} a_i = \sum_{j=0}^{k} b_{2j+1}$, we conclude

$$\left(\sum_{i\in S} a_i\right)^2 = \left(\sum_{j=1}^{k} b_{2j+1}\right)^2 \le \sum_{1\le i\le j\le n} (b_i + \dots + b_j)^2 \le \text{RHS},$$

because each number $(b_i + \dots + b_j)^2$ appear in the right-hand expression.

∇

Problem 94. *Let a, b, c be three non-negative real numbers. Prove that*

$$\frac{a^3}{(a+b)^3} + \frac{b^3}{(b+c)^3} + \frac{c^3}{(c+a)^3} + \frac{5abc}{(a+b)(b+c)(c+a)} \ge 1.$$

(Pham Kim Hung)

SOLUTION. The inequality can be changed into an equivalent form as follow

$$\frac{1}{(1+x)^3} + \frac{1}{(1+y)^3} + \frac{1}{(1+z)^3} + \frac{5}{(1+x)(1+y)(1+z)} \ge 1,$$

where $x = \frac{b}{a}, y = \frac{c}{b}, z = \frac{a}{c}$ and $xyz = 1$. Denote

$$m = 1 - \frac{2}{1+x}, n = 1 - \frac{2}{1+y}, p = 1 - \frac{2}{1+z}.$$

Certainly $m, n, p \in [-1, 1]$ and

$$(1+m)(1+n)(1+p) = (1-m)(1-n)(1-p) \Rightarrow m + n + p + mnp = 0.$$

Our problem becomes

$$\sum_{cyc}(1-m)^3 + 5\prod_{cyc}(1-m) \ge 8$$
$$\Leftrightarrow 3\sum_{cyc} m^2 + 5\sum_{cyc} mn \ge 3\sum_{cyc} m + \sum_{cyc} m^3$$

$$\Leftrightarrow\ 3\sum_{cyc} m^2 + 5\sum_{cyc} mn \geq \sum_{cyc} m^3 - 3mnp.$$

If $mn + np + pm \geq 0$ then LHS ≥ 0. Otherwise, suppose that $mn + np + mp \leq 0$, then LHS $= (m+n+p)^2 - (mn+np+pm) \geq 0$. So, in every case, we have LHS ≥ 0. Moreover, RHS $= (m+n+p)(m^2+n^2+p^2-mn-np-pm)$ has the same sign as $m+n+p$, so we only need to consider the inequality in case RHS ≥ 0 or equivalently $m+n+p \geq 0$. Let $t = m+n+p$ and $u = mn+np+pm$, the inequality becomes

$$3(t^2-2u) + 5u \geq t(t^2-3u) \ \Leftrightarrow\ t^2(3-t) + u(3t-1) \geq 0 \ (\star)$$

According to **AM-GM** inequality, we have

$$m^2+n^2+p^2 \geq 3\left|mnp\right|^{2/3} \geq -3mnp = 3(m+n+p)$$

$$\Rightarrow t^2 - 2u \geq 3t \ \Rightarrow 2u \leq t(t-3) \ (\star\star)$$

If $u \geq 0$ then we immediately have $3\sum_{cyc} m^2 + 5\sum_{cyc} mn \geq 2\sum_{cyc} m^2 \geq \sum_{cyc} m^3 - 3mnp$. Otherwise, suppose that $u \leq 0$. The inequality is also obvious if $3t-1 \leq 0$, so it's enough to consider the case $3t-1 \geq 0$. Replacing $(\star\star)$ to $(\star)$, it remains to prove that

$$2t^2(3-t) + t(t-3)(3t-1) \geq 0 \ \Leftrightarrow\ t(3-t)(1-t) \geq 0 \ \Leftrightarrow\ t(3-t)(1+mnp) \geq 0,$$

which is also obvious because $m, n, p \in [-1, 1]$. Notice that the equality can hold in $(\star)$ for $m = n = p = 0$ and $m = n = 1, p = -1$ up to permutation but the equality holds in the initial inequality just for $a = b = c$.

Comment. With the same approach, we can prove the following inequality

★ *Let a, b, c be three non-negative real numbers. Prove that*

$$\frac{a^2}{(a+b)^2} + \frac{b^2}{(b+c)^2} + \frac{c^2}{(c+a)^2} + \frac{2abc}{(a+b)(b+c)(c+a)} \geq 1.$$

We also have another solution to this problem by **Cauchy-Schwarz** inequality. Indeed, it is equivalent to (after some substitutions)

$$\sum_{cyc} \frac{x^4}{(x^2+yz)^2} + \frac{2x^2y^2z^2}{(x^2+yz)(y^2+zx)(z^2+xy)} \geq 1.$$

By **Cauchy-Schwarz** inequality, we obtain

$$\sum_{cyc} \frac{x^4}{(x^2+yz)^2} \geq \frac{(x^2+y^2+z^2)^2}{(x^2+yz)^2+(y^2+zx)^2+(z^2+xy)^2}.$$

It remains to prove that

$$\frac{(x^2+y^2+z^2)^2}{(x^2+yz)^2+(y^2+zx)^2+(z^2+xy)^2}+\frac{2x^2y^2z^2}{(x^2+yz)(y^2+zx)(z^2+xy)}\geq 1$$

$$\Leftrightarrow \quad \frac{2x^2y^2z^2}{(x^2+yz)(y^2+zx)(z^2+xy)}\geq \frac{x^2y^2+y^2z^2+z^2x^2-x^4+y^4+z^4}{(x^2+yz)^2+(y^2+zx)^2+(z^2+xy)^2}$$

$$\Leftrightarrow \prod_{cyc}(x^2+yz)\left(\sum_{cyc}\frac{1}{x^2}\right)+2\sum_{cyc}(x^2+yz)^2\geq \prod_{cyc}(x^2+yz)\left(\sum_{cyc}\frac{1}{yz}\right).$$

After expanding and reducing similar terms, we obtain an equivalent inequality

$$\sum_{cyc}a^2bc+2\sum_{cyc}a^2b^2+\sum_{cyc}\frac{a^3b^3}{c^2}+\sum_{cyc}\frac{c^4(a^2+b^2)}{ab}\geq 2\sum_{cyc}\frac{a^2b^2(a+b)}{c}+\sum_{cyc}a^3(b+c).$$

Rewrite this inequality in the form $S_a(b-c)^2+S_b(c-a)^2+S_c(a-b)^2$ with

$$S_a=\frac{a^4}{2bc}+\frac{a^3(b^3+c^3)}{b^2c^2}+\frac{(b-c)^2}{2}-\frac{a^2}{2};$$

$$S_b=\frac{b^4}{2ca}+\frac{b^3(c^3+a^3)}{c^2a^2}+\frac{(c-a)^2}{2}-\frac{b^2}{2};$$

$$S_c=\frac{c^4}{2ab}+\frac{c^3(a^3+b^3)}{a^2b^2}+\frac{(a-b)^2}{2}-\frac{c^2}{2};$$

WLOG, suppose that $a\geq b\geq c$, then $S_a,S_b\geq 0$. Moreover,

$$S_b+S_c\geq \frac{b^3(c^3+a^3)}{c^2a^2}-\frac{b^2+c^2}{2}\geq \frac{b^3a}{c^2}-b^2\geq 0.$$

We can conclude that

$$\sum_{cyc}S_a(b-c)^2\geq (S_b+S_c)(a-b)^2\geq 0.$$

∇

Problem 95. *Let a,b,c be arbitrary positive real numbers. Prove that*

$$(a^3+b^3+c^3)^2\geq 2(a^5b+b^5c+c^5a)+abc(a^3+b^3+c^3).$$

(Pham Kim Hung, Le Huu Dien Khue)

SOLUTION. Rewrite the inequality in the following form

$$\sum_{cyc}a^6+2\sum_{cyc}a^3b^3\geq 2\sum_{cyc}a^5b+abc\sum_{cyc}a^3$$

$$\Leftrightarrow\ 2\left(\sum_{cyc} a^6+\sum_{cyc} a^4b^2-2\sum_{cyc} a^5b\right)+\left(\sum_{cyc} a^4b^2+\sum_{cyc} a^4c^2-2abc\sum_{cyc} a^3\right)+$$

$$+\left(2\sum_{cyc} a^3b^3-\sum_{cyc} a^4b^2-\sum_{cyc} a^2b^4\right)\ge\left(\sum_{cyc} a^4b^2-\sum_{cyc} a^4c^2\right)$$

$$\Leftrightarrow\ \sum_{cyc}(2a^4+c^4-a^2b^2)(a-b)^2\ge(a^2-b^2)(b^2-c^2)(a^2-c^2)\ (\star)$$

Denote $M=(a-b)(b-c)(c-a)$. Certainly, we may assume that $a\ge b\ge c$ and $a>c$ in order to have $M\ge 0$. We will prove this inequality using the following results

(1) $$\sum_{cyc}(3a+2c-b)(a-b)^2\ge 4M$$

(2) $$\sum_{cyc}(11a^2+6c^2-b^2-4ab)(a-b)^2\ge 8(a+b+c)M$$

(3) $$\sum_{cyc}(4a^3+2c^3-a^2b-b^2a)(a-b)^2\ge(a^2+b^2+c^2+3ab+3bc+3ca)M$$

There is an interesting relationship between these inequalities: (1) $\Rightarrow$ (2) $\Rightarrow$ (3). We will prove (1) in order to show (2) and then (3).

The proof of (1). Certainly, it suffices to prove the inequality in the case $c=\min\{a,b,c\}=0$ (because if we decrease a,b,c by an arbitrary positive real number smaller then c then the right-hand expression of (1) is unchanged, but the left-hand side is decreased). If $c=0$, the condition (1) becomes

$$(3a-b)(a-b)^2+(3b+2a)b^2+(-a+2b)a^2\ge 3ab(a-b)$$

$$\Leftrightarrow\ 2a^3-8a^2b+10ab^2+2b^3\ge 0\ \Leftrightarrow\ h(x)=2x^3-8x^2+10x+2\ge 0,$$

where $x=\dfrac{a}{b}$. Notice that $h'(x)=2(x-1)(3x-5)$, so it's easy to conclude that

$$h(x)\ge f\left(\frac{5}{3}\right)>5>0.$$

Therefore (1) is proved. Now let us show that (1) gives (2).

The proof of (2). Let $a=a_1+t=f_1(t), b=b_1+t=f_2(t), c=c_1+t=f_3(t)$ then (2) is equivalent to

$$f(t)=\sum_{cyc}\left(11f_1(t)^2-f_2(t)^2-4f_1(t)f_2(t)+6f_3(t)^2\right)(a-b)^2-8(f_1(t)+f_2(t)+f_3(t))M\ge 0.$$

According to (1), we can see that

$$f'(t)=\sum_{cyc}(18f_1(t)-6f_2(t)+12f_3(t))(a-b)^2-24M=6\sum_{cyc}(3a-b+2c)(a-b)^2-24M\ge 0$$

so $f(t) \geq f(-c)$ (because $t \geq -c$). This property shows that it suffices to consider (2) in the case $c = 0$,

$$(11a^2 - b^2 - 4ab)(a-b)^2 + (11b^2 + 6a^2)b^2 + (-a^2 + 6b^2)a^2 \geq 8(a+b)ab(a-b)$$

$$\Leftrightarrow\ 11a^4 - 35a^3b + 30a^2b^2 + 6ab^3 + 10b^4 \geq 0.$$

This inequality is certainly true because **AM-GM** inequality shows that $11a^4 + 30a^2b^2 \geq 2\sqrt{11 \cdot 30}a^3b > 35a^3b$. Therefore (2) is proved successfully.

The proof of (3). Similarly, according to (2) and using the same reasoning as above, we realize that it suffices to prove (3) in case $\min\{a,b,c\} = c = 0$. In this case, the inequality becomes

$$(4a^3 - a^2b)(a-b)^2 + (4b^3 + 2a^3)b^2 + 2b^3a^2 \geq (a^2 + b^2 + 3ab)ab(a-b)$$

$$\Leftrightarrow\ 4a^5 - 10a^4b + 6a^3b^2 + 3a^2b^3 + ab^4 + 4b^5 \geq 0$$

If $2a \geq 3b$ then we are done because $4a^5 - 10a^4b + 6a^3b^2 = 2a^3(a-b)(2a-3b) \geq 0$. Suppose that $2a \leq 3b$ then

$$4a^5 - 10a^4b + 6a^3b^2 + 3a^2b^3 \geq 4a^5 - 10a^4b + 8a^3b^2 \geq (2\sqrt{32} - 10)a^4b \geq 0.$$

Are this step, (3) is proved successfully.

The proof of $(\star)$. Similarly, according to (3) and using the same reasoning, we may assume that $c = 0$. The inequality becomes simple as $(a^3 + b^3)^2 \geq 2a^5b$ or $g(a) = a^6 - 2a^5b + 2a^3b^3 + b^6$. It's easy to check that $g'(a) \geq 0$, thus $g(a) \geq g(b) \geq 0$. The inequality is completely proved and the equality holds for $a = b = c$.

Comment. The solution above is based on the mixing variable method. This problem can help us prove a very hard inequality, proposed by an anonymous, as follows

★ *Let a, b, c be positive real numbers such that $abc = 1$. Prove that*

$$\frac{a}{b^4+2} + \frac{b}{c^4+2} + \frac{c}{a^4+2} \geq 1.$$

To prove it, we denote $a = \dfrac{y}{x}, b = \dfrac{z}{y}, c = \dfrac{x}{z}$, then by **Cauchy-Schwarz** inequality and the previous result, we can conclude that

$$\sum_{cyc} \frac{a}{b^4+2} = \sum_{cyc} \frac{y^5}{xz^4 + 2xy^4} = \sum_{cyc} \frac{y^6}{xyz^4 + 2xy^5}$$

$$\geq \frac{(x^3+y^3+z^3)^2}{xyz(x^3+y^3+z^3) + 2(xy^5 + yz^5 + zx^5)} \geq 1.$$

∇

Problem 96. *Let a, b, c, d be non-negative real numbers such that*

$$(a+b+c+d)^2 = 3(a^2+b^2+c^2+d^2).$$

Prove the following inequality

$$a^4+b^4+c^4+d^4 \geq 28abcd.$$

(Pham Kim Hung)

SOLUTION. Let $m = a+b, n = c+d, x = ab$ and $y = cd$ then $(m+n)^2 = 3(m^2+n^2-2x-2y)$ or $3(x+y) = m^2+n^2-mn$. The problem becomes

$$F = 2(x^2+y^2) - 4(m^2x+n^2y) - 28xy + m^4+n^4 \geq 0.$$

Now we fix m, n (as constants) and let x, y vary (as variables) such that $x \leq \frac{m^2}{4}, y \leq \frac{n^2}{4}$ and $x+y = s = \frac{m^2+n^2-mn}{3}$. In this case, F can be rewritten as

$$F = 2s^2 + m^4 + n^4 - 4(m^2x+n^2y+8xy)$$

Since

$$m^2x+n^2y+8xy = m^2x+n^2(s-x)+8x(s-x) = -8x^2+(m^2-n^2+8s)x+n^2s = f(x)$$

is a concave function of x, it follows that the maximum value of $f(x)$ is attained if an only if $x = \frac{m^2}{4}$ (as an upper bound of x) or $x = \frac{m^2-n^2+8s}{16}$ (as the unique root of the function $f'(x)$, if it exists). From this, the problem can be divided into two smaller cases

(i). The first case. If $x = \frac{m^2-n^2+8s}{16}$ then $y = \frac{-m^2+n^2+8s}{16}$. Let $\alpha = m^2+n^2$ and $\beta = mn$ then

$$f(x) = \frac{(m^2-n^2+8s)^2+32n^2s}{32} = \frac{\alpha^2+16s\alpha-4\beta^2+64s^2}{32}.$$

We need to prove that

$$16s^2+8\alpha^2-16\beta^2 \geq \alpha^2+16s\alpha-4\beta^2+64s^2$$

$$\Leftrightarrow\ 7\alpha^2 \geq 48s^2+16s\alpha+12\beta^2$$

$$\Leftrightarrow\ 21\alpha^2 \geq 16(\alpha-\beta)^2+16\alpha(\alpha-\beta)+36\beta^2$$

$$\Leftrightarrow\ 21\alpha^2 \geq 32\alpha^2-48\alpha\beta+52\beta^2$$

$$\Leftrightarrow\ -11\alpha^2+48\alpha\beta-52\beta^2\geq 0\ \Leftrightarrow\ (-11\alpha+26\beta)(\alpha-2\beta)\geq 0.$$

Notice that $\alpha-2\beta=(m-n)^2\geq 0$. So it suffices to prove that

$$11\alpha\leq 26\beta\ \Leftrightarrow\ 11(m^2+n^2)\leq 26mn\ \Leftrightarrow\ \frac{13-\sqrt{48}}{11}\leq\frac{m}{n}\leq\frac{13+\sqrt{48}}{11}\ (\star)$$

Because $x\leq\dfrac{m^2}{4}$ and $y\leq\dfrac{n^2}{4}$, we must have

$$\begin{cases}\dfrac{m^2-n^2+8s}{16}\leq\dfrac{m^2}{4}\Rightarrow 8s\geq 3m^2+n^2\Rightarrow 5n^2\leq 8mn+m^2\Rightarrow\dfrac{n}{m}\leq\dfrac{4+\sqrt{20}}{5}.\\ \dfrac{n^2-m^2+8s}{16}\leq\dfrac{n^2}{4}\Rightarrow 8s\geq 3n^2+m^2\Rightarrow 5m^2\leq 8mn+n^2\Rightarrow\dfrac{m}{n}\leq\dfrac{4+\sqrt{20}}{5}.\end{cases}$$

These results confirm condition $(\star)$ immediately and the proof is finished.

(i). The second case. If $x=\dfrac{m^2}{4}$ then $y=\dfrac{4s-m^2}{4}=\dfrac{m^2+4n^2-4mn}{12}$. We need to prove that

$$2s^2+m^4+n^4\geq 4\left(\frac{m^4}{4}+\frac{(m^2+4n^2-4mn)n^2}{12}+\frac{m^2(m^2+4n^2-4mn)}{6}\right)$$

$$\Leftrightarrow\ 2(m^2-mn+n^2)^2+9n^4\geq 3(m^2n^2+4n^4-4mn^3)+6(m^4+4m^2n^2-4m^3n)$$

$$\Leftrightarrow\ -4m^4+20m^3n-21m^2n^2+8mn^3-n^4\geq 0$$

$$\Leftrightarrow\ (2m-n)^2(-m^2+4mn-n^2)\geq 0.$$

Because $y=\dfrac{m^2+4n^2-3mn}{12}\leq\dfrac{n^2}{4}$, it follows that $m^2+n^2\leq 3mn$ and therefore the inequality is obviously true. The proof of this case is finished.

We have the desired result. The equality holds for unexpected cases $(a,b,c,d)\sim(3,1,1,1)$ or $(a,b,c,d)\sim(2+\sqrt{3},2+\sqrt{3},2-\sqrt{3},2-\sqrt{3})$ up to permutation.

∇

Problem 97. *Let a,b,c be positive real numbers with sum 3. Prove that*

$$\frac{a}{b^2+c}+\frac{b}{c^2+a}+\frac{c}{a^2+b}\geq\frac{3}{2}.$$

(Pham Kim Hung)

SOLUTION. After expanding, we can change the inequality into

$$2\sum_{cyc}a^4+2\sum_{cyc}a^2b+3abc\geq 3a^2b^2c^2+\sum_{cyc}a^3b^2+3\sum_{cyc}ab^3$$

Let $M = ab + bc + ca$ and $S = (a-b)(a-c)(b-c)$. According to the identities

$$2\sum_{cyc} a^2b = S + 3M - 3abc\ ;$$
$$2\sum_{cyc} a^3b^2 = SM + 3\sum a^2b^2 - Mabc\ ;$$
$$2\sum_{cyc} ab^3 = \sum a^3(b+c) - 3S\ ;$$

the inequality is equivalent to

$$4\sum_{cyc} a^4 + 11S + 6M + Mabc \geq 6a^2b^2c^2 + SM + 3\sum_{cyc} a^2b^2 + 3\sum_{cyc} a^3(b+c).$$

Notice that $abc(M+3) \geq 6a^2b^2c^2$, so we need to prove that $A \geq S(M-11)$ where

$$A = 4\sum_{cyc} a^4 + 6M - 3\sum_{cyc} a^2b^2 - 3\sum_{cyc} a^3(b+c) - 3abc$$

Represent A with the help of some squares as

$$3A = 12\sum_{cyc} a^4 + 7abc\sum_{cyc} a - 5\sum a^2b^2 - 7\sum a^3(b+c)$$

$$\Rightarrow 6A = \sum_{cyc}(12a^2 + 12b^2 + 10ab - 7c^2)(a-b)^2.$$

Because $M \leq 11$, we may assume that $S \leq 0$ and $b \geq a \geq c$. We will prove

$$6A \geq -66S \Leftrightarrow 6A \geq 22(a+b+c)(a-b)(b-c)(c-a)\ (\star)$$

If $\min(a,b,c) = c = 0$ then the problem is proved because

$$6A = \sum(12a^2 + 12b^2 + 10ab - 7c^2)(a-b)^2$$
$$= (12a^2 + 12b^2 + 10ab)(a-b)^2 + (12a^2 - 7b^2)a^2 + (12b^2 - 7a^2)b^2$$
$$= 10a^2b^2 + (24a^2 + 24b^2 + 34ab)(a-b)^2$$
$$\geq 10a^2b^2 + \frac{41}{2}(a^2-b^2)^2 \geq 2\sqrt{5.41}ab(b^2-a^2) > 22ab(a-b)(a+b).$$

Now suppose that $\min(a,b,c) > 0$. Since $(\star)$ is homogeneous, we can dismiss the condition $a+b+c=3$ but prove $(\star)$ for arbitrary positive numbers a, b, c. We realize first that if we replace a, b, c with $a+t, b+t, c+t$ then the difference between the two sides is increased. Indeed, we will prove that

$$\sum_{cyc}(12(a+t)^2+12(b+t)^2+10(a+t)(b+t)-7(c+t)^2-12(a^2+b^2)-10ab+7c^2)(a-b)^2 \geq 66tS,$$

which is inferred from

$$\sum_{cyc}(17a+17b-7c)(a-b)^2 \geq 33(a-b)(b-c)(c-a) \ (\star\star)$$

Using the same method one more time (replacing a, b, c with $a+t, b+t, c+t$), we only need to examine $(\star\star)$ in case $\min(a,b,c) = c = 0$, or equivalently

$$17(a+b)(a-b)^2 + (17a-7b)a^2 + (17b-7a)b^2 \geq 22ab(b-a),$$

which is obviously true because

$$\text{LHS} \geq 17(a+b)(a-b)^2 + 10(a^3+b^3) \geq 2\sqrt{17.10(a+b)(a^3+b^3)}(b-a) \geq \text{RHS}.$$

This last step completes the proof. The equality holds for $a = b = c = 1$.

∇

Problem 98. *Let a, b, c be positive real numbers. Prove that*

$$\frac{1}{(2a+b)^2} + \frac{1}{(2b+c)^2} + \frac{1}{(2c+a)^2} \geq \frac{1}{ab+bc+ca}.$$

(Pham Kim Hung)

SOLUTION. After expanding, the inequality becomes

$$\sum_{cyc}(4a^5b + 4a^5c - 12a^4b^2 + 12a^4c^2 + 5a^3b^3 + +8a^4bc - 19a^3b^2c + 5a^3c^2b - 7a^2b^2c^2) \geq 0$$

or

$$6\sum_{cyc} ab(a^2-b^2-2ab+2ac)^2 + \sum_{sym}(2a^5b - a^3b^3 - 4a^4bc + 10a^3b^2c - 7a^2b^2c^2) \geq 0.$$

It remains to prove that

$$2\sum_{cyc} a^5(b+c) - 2\sum_{cyc} a^3b^3 - 8\sum_{cyc} a^4bc + 10abc\sum_{cyc} ab(a+b) - 42a^2b^2c^2 \geq 0.$$

Using the identity $2(a-b)^2(b-c)^2(c-a)^2 \geq 0$, we infer that

$$2\sum_{cyc} a^4(b^2+c^2) + 4abc\sum_{cyc} a^2(b+c) - 4\sum_{cyc} a^3b^3 - 12a^2b^2c^2 - 4abc\sum_{cyc} a^3 \geq 0.$$

Finally, we have to prove that

$$2\sum_{cyc} a^5(b+c) + 6abc\sum_{cyc} ab(a+b) + 2\sum_{cyc} a^3b^3 \geq 4abc\sum_{cyc} a^3 + 30a^2b^2c^2 + 2\sum_{cyc} a^4(b^2+c^2).$$

This inequality can be rewritten using square

$$S_a(b-c)^2 + S_b(c-a)^2 + S_c(a-b)^2 \geq 0,$$

where

$$S_a = 2bc(b^2+bc+c^2) + a^3(b+c) - 2abc(b+c) + 6a^2bc\ ;$$

$$S_b = 2ca(c^2+ca+a^2) + b^3(c+a) - 2abc(c+a) + 6c^2ab\ ;$$

$$S_c = 2ab(a^2+ab+b^2) + c^3(a+b) - 2abc(a+b) + 6c^2bc\ ;$$

Clearly, S_a is non-negative since (applying **AM-GM** inequality)

$$S_a \geq bc(b+c)^2 + 6a^2bc - 2abc(b+c) \geq 2\left(\sqrt{6}-1\right)abc(b+c) \geq 0.$$

Similarly, S_b and S_c are non-negative, so the inequality is completely proved, and we are done. The equality holds only for $a = b = c$.

∇

Problem 99. *Suppose that $x_1 \geq x_2 \geq ... \geq x_{2n-1} \geq x_{2n} \geq 0$ are real numbers and $x_1 + x_2 + ... + x_{2n} = 2n-1$. Find the maximum of the following expression*

$$P = \left(x_1^2 + x_2^2\right)\left(x_3^2 + x_4^2\right)...\left(x_{2n-1}^2 + x_{2n}^2\right).$$

(Pham Kim Hung)

SOLUTION. Although it's very hard to solve this problem directly, we find out unexpectedly that proving the general problem is simpler. In fact, the proposed problem is a direct corollary of the following general result

★ *Suppose that $\epsilon \leq \frac{k}{2n}$ is a positive constant and $x_1 \geq x_2 \geq ... \geq x_{2n-1} \geq x_{2n} \geq \epsilon \geq 0$ are real numbers satisfying that $x_1 + x_2 + ... + x_{2n} = k = const$. The expression*

$$P_n = \left(x_1^2 + x_2^2\right)\left(x_3^2 + x_4^2\right)...\left(x_{2n-1}^2 + x_{2n}^2\right)$$

attains the maximum if and only if $x_1 = ... = x_{2n-1}$ and $x_{2n} = \epsilon$.

We will prove this general result by the inductive method. Before performing induction steps, we figure out three results (they are built deliberately, not accidentally, according to how the induction step progresses.

Lemma 1. *Let $x \geq y \geq z \geq t \geq 0$ and $y + z = 2\alpha$ then*

$$(x^2+y^2)(z^2+t^2) \leq (x^2+\alpha^2)(\alpha^2+t^2).$$

PROOF. Let $y = \alpha + \beta$ and $z = \alpha - \beta$, $\beta \geq 0$, then $x \geq \alpha + \beta \geq \alpha - \beta \geq t$. Denote

$$f(\beta) = \left[x^2 + (\alpha+\beta)^2\right]\left[(\alpha-\beta)^2 + t^2\right]$$

then it's enough to prove that $f'(\beta) \leq 0$, which is clearly true because

$$f'(\beta) = -2x^2(\alpha-\beta) + 2t^2(\alpha+\beta) - 2\beta(\alpha^2-\beta^2) \leq -2x^2.t + 2t^2.x \leq 0.$$

Lemma 2. *Let $x \geq y \geq z \geq 0$ and $(2n-1)x + 2y = (2n+1)\gamma$ ($n \in \mathbb{N}$, $n \geq 2$) then*

$$x^{2n-2}(x^2+y^2)(y^2+z^2) \leq 2\gamma^{2n}(\gamma^2+z^2).$$

PROOF. There exists a real number $\beta \geq 0$ for which $x = \gamma + 2\beta$ and $y = \gamma - (2n-1)\beta$. Thus, we must have $\gamma - (2n-1)\beta \geq z$. Denote

$$g(\beta) = (\gamma+2\beta)^{2n}(\gamma-(2n-1)\beta)^2 + (\gamma+2\beta)^{2n-2}(\gamma-(2n-1)\beta)^4 \\ + (\gamma+2\beta)^{2n}z^2 + (\gamma+2\beta)^{2n-2}(\gamma-(2n-1)\beta)^2 z^2.$$

Clearly $g(\beta) = x^{2n-2}(x^2+y^2)(y^2+z^2)$ and we need to prove that $g'(\beta) \leq 0$. Indeed (in the expression of $g'(\beta)$, we denote $x = \gamma + 2\beta$ and $y = \gamma - (2n-1)\beta$ for a shorter presentation, but we still think of them as related to the variable γ)

$$g'(\beta) = 4nx^{2n-1}y^2 - (4n-2)x^{2n}y + (4n-4)x^{2n-3}y^4 - (8n-4)x^{2n-2}y^3 \\ + 4nx^{2n-1}z^2 + (4n-4)y^{2n-3}x^2z^2 - (4n-2)x^{2n-2}yz^2$$

Because $x \geq y \geq z$, we infer that

$$4nx^{2n-1}z^2 + (4n-4)y^{2n-3}x^2z^2 - (4n-2)x^{2n-2}yz^2$$

$$\leq 4nx^{2n-1}z^2 - 2x^{2n-2}yz^2 \leq 4nx^{2n-1}y^2 - 2x^{2n-2}y^3.$$

It suffices to prove that

$$8nx^{2n-1}y^2 + (4n-4)x^{2n-3}y^4 \leq (4n-2)x^{2n}y + (8n-2)x^{2n-2}y^3$$

$$\Leftrightarrow\ 4nx^2y + (2n-2)y^3 \leq (2n-1)x^3 + (4n-1)xy^2$$

$$\Leftrightarrow\ (x-y)\left[(2n-1)x^2 - (2n+1)xy + (2n-2)y^2\right] \geq 0,$$

which is clearly true because $x \geq y$. This ends the proof of the second lemma.

We also notice that lemma 2 is still true for $n = 1$ (the solution if $n = 1$ is much simpler than the proof for the general case $n \geq 2$, so I will not show it here).